Table of Contents

STATICS

"Statics" is typically the first course to which structural engineering students are exposed. In this course, the structures are simple enough that all relevant forces can be determined without knowledge of the physical materials that the structures are made of. In fact, the structures in this course can be assumed rigid, even though no real life structures are truly rigid. By the end of the course, the students should be able to apply the equations of static "equilibrium" to *any* structure, by creating "free-body-diagrams." A carefully chosen free-body-diagram is the starting point when analyzing any real life structure, whether it is an entire building, a component of a building, or a machine, biological structure, *etc*. By solving many statics problems, students may begin to develop an intuition for the "path" that forces tend to take within certain kinds of structures. More importantly, as we will see in later outlines, a carefully chosen free-body-diagram is almost always the starting point when deriving important structural engineering principles beyond "statics."

Force equilibrium

For an object not to be accelerating, the following *must be* satisfied:

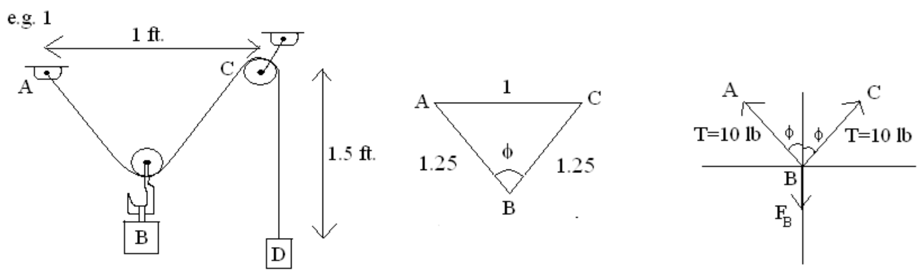
$$\sum Fx = 0$$

$$\sum Fy = 0$$

e.g. 1

Given: The total length of cord is 4 feet. $\overline{CD}$ =1.5 ft. $\overline{AC}$ =1 ft. D=10lb. Ignore mass and size of pulleys.

Find: Weight of B



$$\overline{AB} = \overline{BC} = \frac{4-1.5}{2} = \textit{1.25 ft.}$$

When 3 sides of a triangle are known: $\cos\phi = \dfrac{1.25^2 + 1.25^2 - 1^2}{2(1.25)(1.25)} = .68$

$$\Rightarrow \phi = \cos^{-1} .68 = 47°$$

$$\theta = \frac{47°}{2} = 23.5°$$

Tension=T=10 lb

$$\xrightarrow[+]{} \sum Fx : 10\sin 23.5° - 10\sin 23.5° = 0$$

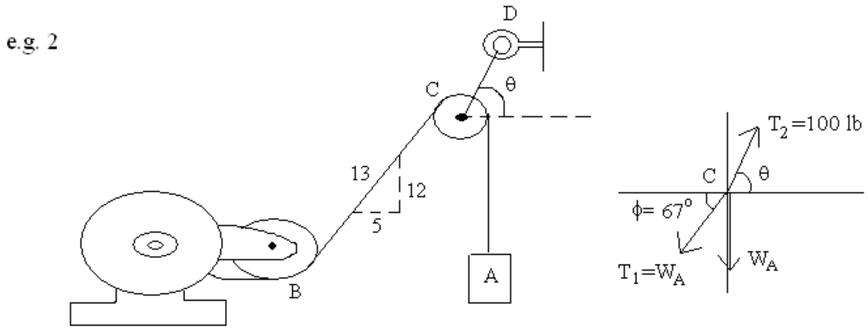
$$+\uparrow \sum Fy : 10\cos 23.5° + 10\cos 23.5° - F_B = 0 \Rightarrow F_B = \boldsymbol{18.3\ lb}$$

note: if F_B had turned out to be a negative value, then our assumed direction on the far right diagram would need to be reversed.

e.g. 2

Given: Crate A is to be hoisted at constant velocity (∴ this is a statics problem). Max tension in both ropes is 100 lb.

Find: θ *and* $(W_A)max$



$$cos\phi = \frac{5}{13} \Rightarrow \phi = 67°$$

$$\xrightarrow[+]{} \sum Fx : 100\cos\theta - T_1\cos 67° = 0$$
$$+\uparrow \sum Fy : 100\sin\theta - T_1\sin 67° - W_A = 0$$

But, $T_1 = W_A$

From $\sum Fx = 0$, $cos\theta = W_A cos67° \div 100 \Rightarrow \theta = \cos^{-1}(W_A cos67° \div 100)$

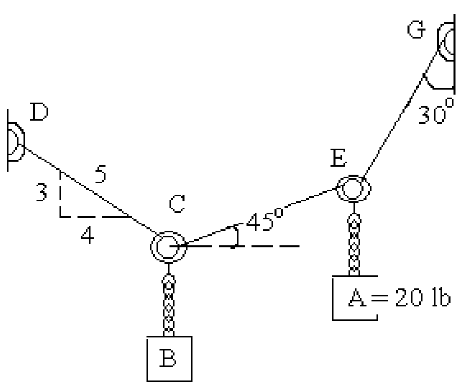
Substitute θ *into* $\sum Fy$ *:*

$100sin[\cos^{-1}(W_A\, cos67° \div 100)] - W_A\, sin67° - W_A = 0 \Rightarrow W_A = \mathbf{51\ lb}$

$\theta = \mathbf{78.5°}$

note: If W_A turned out to be > 100, then we would need to redo the calculation, this time setting W_A=100 and solving for T_2. This would then yield the correct value of θ and T_2.

e.g. 3



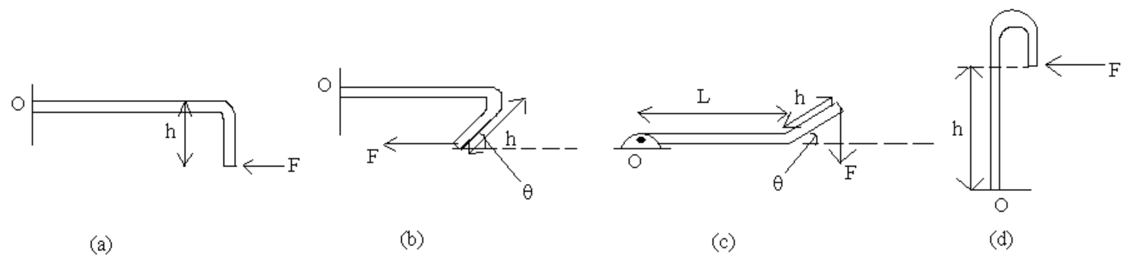
Draw a free body diagram at E and solve for T_{EC} *and* T_{EG}*. Then, draw a free body diagram at C and solve for* T_{CD} *and* W_B*.*

note: Resultants in three dimensional space are easily found as well, using
$\sum Fx = \sum Fy = \sum Fz = 0$

note: Methods that utilize equilibrium are generally preferred in engineering, whereas vector methods such as the parallelogram law and the "dot product" are typically used to illustrate math concepts.

Moments

e.g.'s

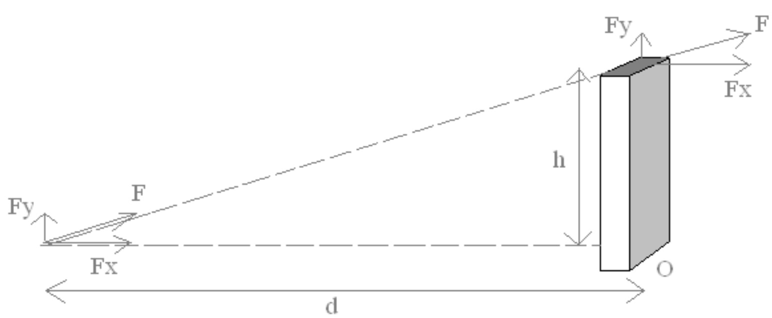


Magnitude of moments about pivot O:

*(a) F*h* *(b) F*hsinθ* *(c) F*(L+hcosθ)* *(d) F*h*

note: the "cross product" can be used as well to determine the moment
The following examples essentially use the distributive law of cross products.

e.g. 1

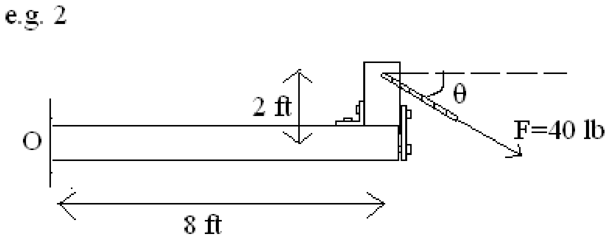


$M_o = Fx*h$

or

$M_o = Fy*d$

e.g. 2
Given: Length and height of lever, magnitude of force.
Find: $\theta_{max,}$ which yields the maximum moment about O, and $\theta_{min,}$ which yields the minimum moment.



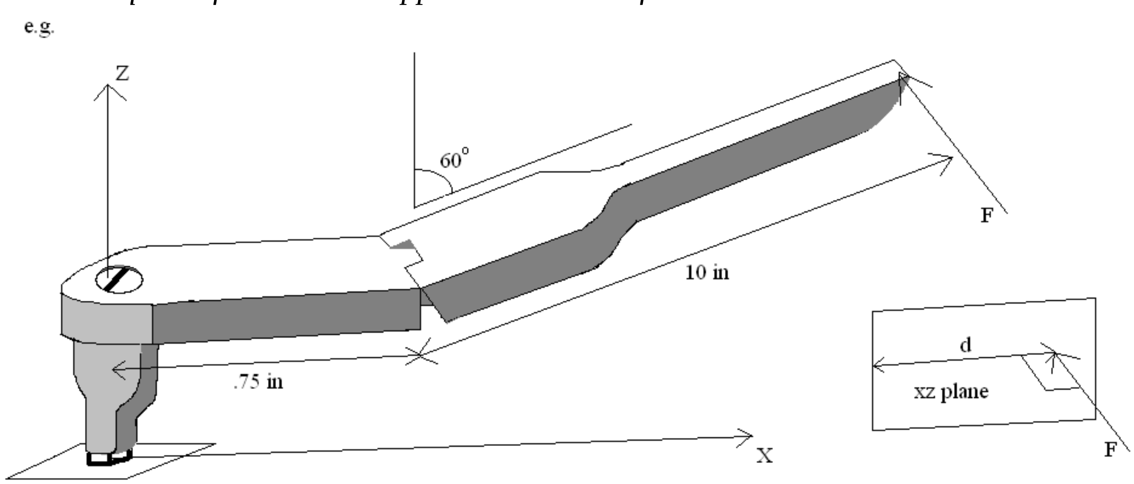
$M_o = (40\cos\theta)(2)+(40\sin\theta)(8) =$
$80\cos\theta+320\sin\theta = 0 = (M_o)_{min} \Rightarrow$
$\theta_{min}=2.90rad \ x \ \frac{180\deg}{\pi \cdot rad}$ **=166°**
note: $\tan^{-1}\frac{2\,ft}{8\,ft} = 14°$ *and 180°-14°=166°*
which is what we found above.

In other words, if the line of action of the force passes through point O, then there is no moment (M_o=0). This, of course, makes intuitive sense.

$M_o' = -80\sin\theta+320\cos\theta = 0 \Rightarrow$
$\theta_{max} = 1.326rad \ x \ \frac{180\deg}{\pi \cdot rad}$ **= 76°**$\Rightarrow$
$Mo = 330\ lb = (M_o)_{max}$

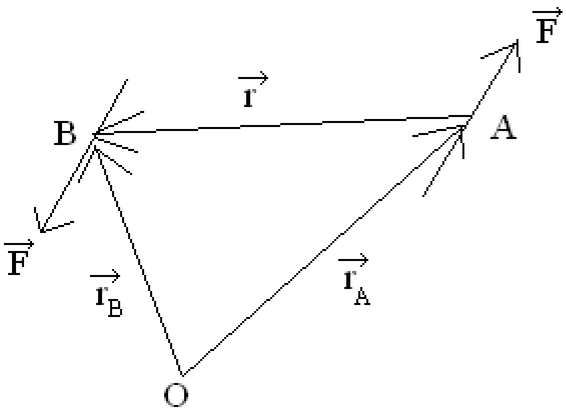
note: the following example could be solved using the "triple scalar product". However, suffice to say, it is really the same as previous problems if the axes are rotated so that the z axis points into the page.

e.g.
*Given: Torque of 80 lb*in required to loosen the nut (M_z=80).*
Dimensions and orientation of flex-head wrench given.
Find: Required force F to be applied to the end of the wrench.



$M_z = Fd$
$d = .75 + 10\sin 60°$
$80 = F*(.75+10\sin 60°) \Rightarrow F = \mathbf{8.5\ lb}$

Moment couple

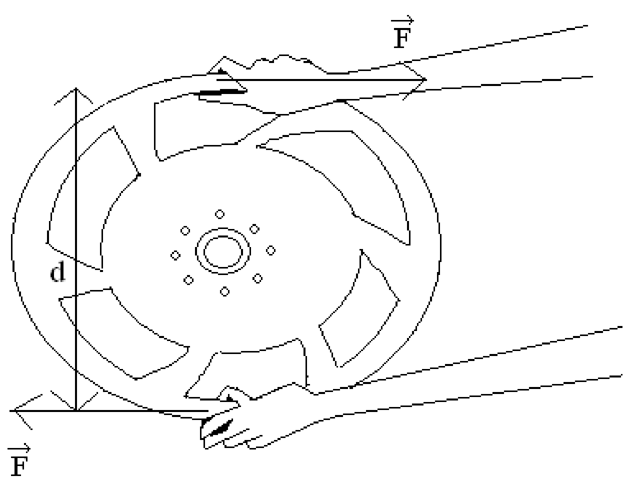


O is an arbitrary point in space
The two $\vec{F}$ forces are located in the same plane and are of equal magnitude.

$\overrightarrow{Mo} = \vec{r}_A \times (-\vec{F}) + \vec{r}_B \times \vec{F}$ (summation of cross products)
$\overrightarrow{Mo} = (\vec{r}_B - \vec{r}_A) \times \vec{F}$
Since $(\vec{r}_B - \vec{r}_A) = \vec{r}$,
$\overrightarrow{Mo} = \vec{r} \times \vec{F}$ where $\vec{F}$ is one of the forces and $\vec{r}$ is the distance between the couple.

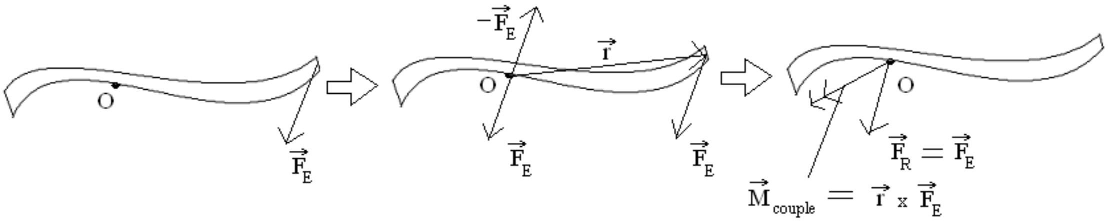
For most problems, $\vec{r}$ can be expressed simply as d, the perpendicular distance between the couple.

$$M_o = Fd$$



The shaft pictured below could be *anywhere* on the wheel and would have the same moment about its axis (as long as it is perpendicular to the wheel). The moment would be less than Fd for an axis not perpendicular to the wheel.

Equivalent force for rigid body



By looking at the above diagram from right to left, we can see that a force and a moment can be reduced to a single force located at a certain distance.

If there are multiple forces, then $F_{Rx} = \sum Fx$ and $F_{Ry} = \sum Fy$.

$\vec{F}_R = < F_{Rx}, F_{Ry} >$

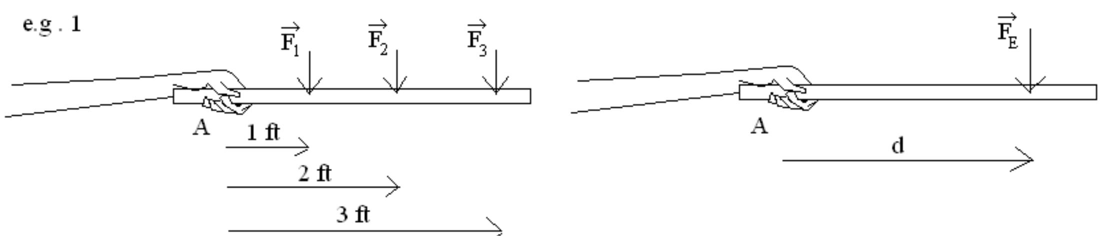
$\vec{M}_{Ro} = \sum Mo$

note: the magnitude and direction of $\vec{F}_R$ is independent of the location of O. But, $\vec{M}_{Ro}$ depends upon the position vectors $\vec{r}$ and therefore depends upon the location of O.

$\vec{F}_E$ would have the same magnitude as $\vec{F}_R$ but would be located at a distance $d = \vec{M}_{Ro} \div \vec{F}_R$, a distance from the chosen point O.

note: $\vec{F}_E$ and its line of action is the only possible $\vec{F}_E$ for a given system. *Where* you take the moment determines the magnitude and location of d, but it always puts $\vec{F}_E$ on the same line of action.

e.g. 1
Given: $F_1 = 2$ lb, $F_2 = 1$ lb, $F_3 = 3$ lb
Find: equivalent force F_E and location d of this force
.



$M_A = 2*1+1*2+3*3 = 13$ *lb*ft*

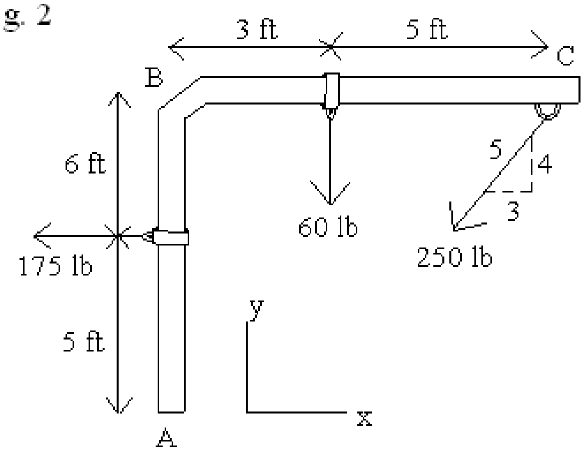
(this moment would be resisted by the person's fingers and palm of hand)

$F_E = F_R = 2+1+3 =$ ***6 lb***

$F_E * d = M_A \Rightarrow d = \frac{13}{6} = \boldsymbol{2.1667\ ft}$

e.g. 2
Given: Three forces acting at the locations shown.
Find: Distance from A to the point where the line of action of $\vec{F}_E$ intersects $\overline{AB}$.
Distance from B to the point where the line of action of $\vec{F}_E$ intersects $\overline{BC}$.

e.g. 2

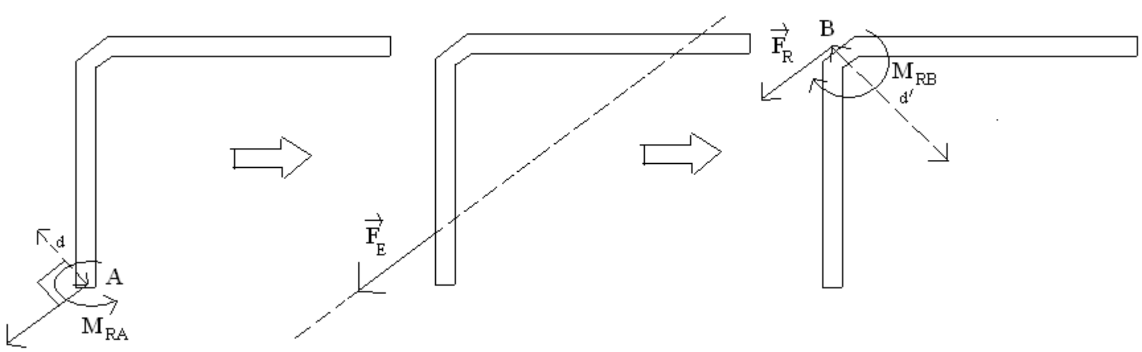


$$\sum Fx = -175 - 250 * \frac{3}{5} = -325 lb$$

$$\sum Fy = -60 - 250 * \frac{4}{5} = -260 lb$$

$$|F_R| = \sqrt{325^2 + 260^2} = 416 lb$$

note:



$$\sum M_{RA} = 175*5 + 250*\frac{3}{5}*(6+5) - 60(3) - 250*\frac{4}{5}*(5+3) = 745 lb * ft$$

$$d = \frac{745}{416} = 1.79\ ft$$

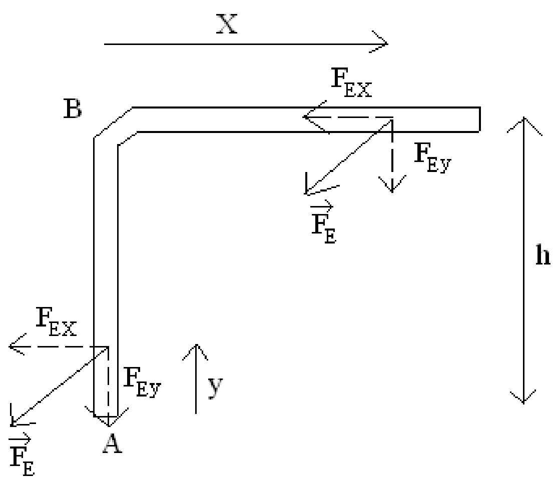
OR

$$\sum M_{RB} = -175*6-60(3)-250*\frac{4}{5}*(3+5) = -2830 lb*ft$$

$$d' = \frac{-2830}{416} = -6.8\,ft$$

Both solutions seem to place $\vec{F}_E$ *along our expected line of action, as the above illustration shows.*

Going back to the original question and solution $\rightarrow$



Looking first at the lower left portion:

$F_{EX}*y + F_{EY}*0 = 745$ *(found from* $\sum M_{RA}$*)*

$\Rightarrow y = \mathbf{2.29\ ft}$

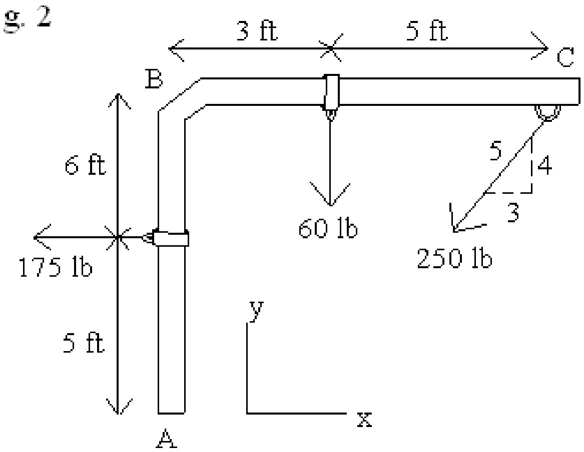
Now looking at the upper right portion:

$F_{EX}*h - F_{EY}*x = 2830 \Rightarrow x = \mathbf{10.9\ ft}$

Equilibrium of a rigid body

So far, we've looked at types of problems which are mostly academic. Practical engineering problems involve taking applied forces and using engineering methods to calculate how those applied forces are distributed among the elements of your structure. This was discussed in the first chapter, titled "Introduction to Structural Engineering." Now that we have the needed foundation, we will begin to look at more practical-type engineering problems.

e.g. 2



Previously, we've seen that $\vec{F}_E$ creates a different moment depending on the location of O. In the previous example, shown to the right, we summed moments about point A to find the location of $\vec{F}_E$. This moment is resisted by the connection at A, otherwise the whole structure would clearly tip over. In the example pictured to the right, there are actually three "support reactions" at point A. There are two forces, along with the resisting moment described two sentences ago. These three support reactions combine to create a vector that is exactly equal and opposite $\vec{F}_E$. Thus, the structure as a whole is static. So, we have three unknowns at point A, and we need three equations.

$\sum Fx = 0$ O may lie either on or off the body and can be a point or an axis.

$\sum Fy = 0$ OR, $\sum M_A = 0$ A,B, and C must not lie on the same line.

$\sum Mo = 0$ $\quad \sum M_B = 0$

$\sum M_C = 0$

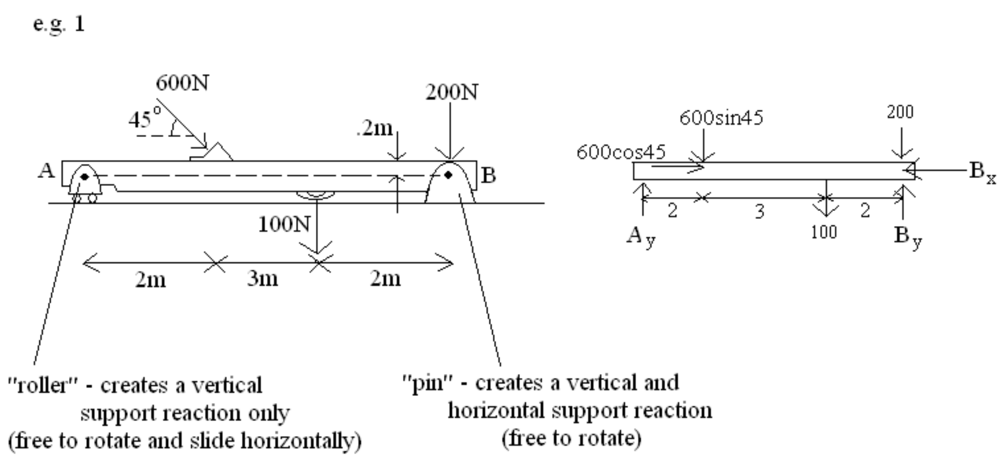
Alternatively,

$\sum F_a = 0$ A line passing through points A and B must NOT be perpendicular to the "a" axis. If they were allowed to be perpendicular, then there

$\sum M_A = 0$ could be $\vec{F}_R \neq 0$ which is perpendicular to the "a" axis ($\sum F_a = 0$)

$\sum M_B = 0$ and has a line of action along $\overline{AB}$ ($\sum M_A = \sum M_B = 0$). To prevent this, the condition can simply be stated that $\overline{AB}$ cannot be perpendicular to the "a" axis.

e.g. 1
Find: Support reactions (see pic below)



$$\sum Fx: 600\cos 45 - Bx = 0 \Rightarrow Bx = \boldsymbol{424N}$$

$$\sum M_B: 100(2) + (600\sin 45)(5) - Ay(7) - (600\cos 45)(.2) = 0 \Rightarrow Ay = \boldsymbol{319N}$$

$$\sum Fy: Ay - 600\sin 45 - 100 - 200 + By = 0 \Rightarrow By = \boldsymbol{405N}$$

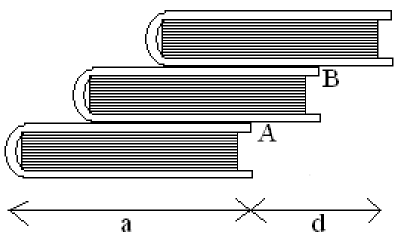
OR

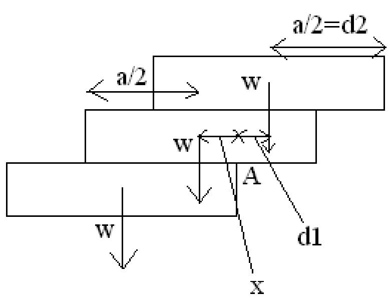
$$\sum M_A: By(7) - (600\sin 45)(2) - (600\cos 45)(.2) - 100(5) - 200(7) = 0 \Rightarrow By = 405N$$

note: $\overline{AB}$ not perpendicular to the x axis $\Rightarrow$ $\begin{matrix}\sum Fx = 0 \\ \sum M_B = 0 \\ \sum M_A = 0\end{matrix}$ can be used.

e.g.2
Given: The weight of each book is w and each has length "a".
Find: Maximum distance d that the top book can extend over the bottom book, before the stack topples. (see pic below)

e.g. 2





Half of the book's length can hang out over B. Need to find the distance that the second book can hang over A.

$$d_2 + x + d_1 = \frac{a}{2} + x + d_1 = a \;\Rightarrow\; x = a - \frac{a}{2} - d_1 = \frac{a}{2} - d_1$$

[normal(contact) forces not shown]

tip over $\Rightarrow M_{top} = M_{mid}$

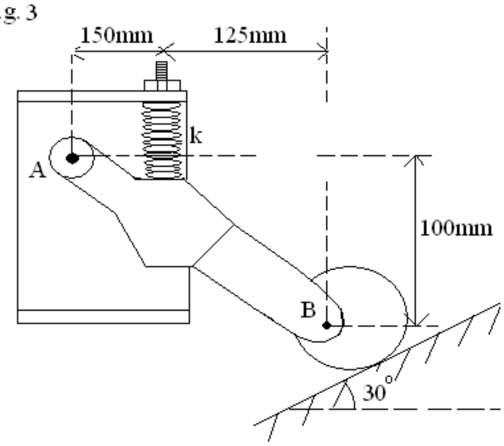
$$w_{middle}(x) - w_{top}(d_1) = 0$$

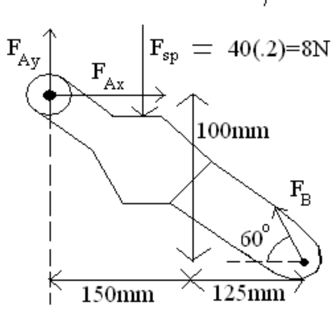
$$w(\frac{a}{2} - d_1) - w(d_1) = 0$$

$$\frac{a}{2} = 2d_1 \Rightarrow d_1 = \frac{a}{4} \Rightarrow d = d_1 + d_2 = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

e.g. 3
Given: (see pic) Spring constant k = 40N/m. Spring is compressed .2m. Angle of slope 30° as shown.
Find: Support reactions at A in terms of F_{Ax} and F_{Ay}, and magnitude of resultant force at wheel bearing B.

e.g. 3





$$\xrightarrow{+} \sum Fx : F_{Ax} - F_B \cos 60° = 0$$

$$+\uparrow \sum Fy : F_{Ay} + F_B \sin 60° - 8 = 0$$

$$\circlearrowleft + \sum M_B : 8(.125) - F_{Ay}(.275) - F_{Ax}(.1) = 0$$

From $\sum Fx$, *$F_{Ax} = F_B cos60°$*

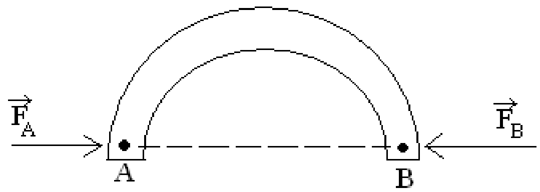
From $\sum Fy$, *$F_{Ay} = 8 - F_B sin60°$*

Substitute into $\sum M_B \Rightarrow$

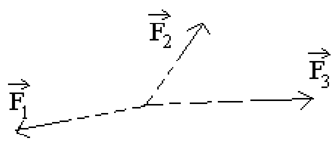
F_B = ***6.38N*** $\quad F_{Ax}$ = ***3.19N*** $\quad F_{Ay}$ = ***2.47N***

Two and Three Force Members

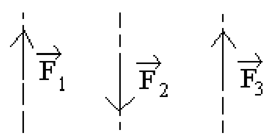
When forces are applied at only two points on a member, $\vec{F}_A$ *must* be of equal magnitude ad opposite direction to $\vec{F}_B$ (see pic). The line of action is along $\overline{AB}$.



If a member is subjected to forces at only three points, then $\vec{F}_1, \vec{F}_2,$ and $\vec{F}_3$ must be either concurrent:



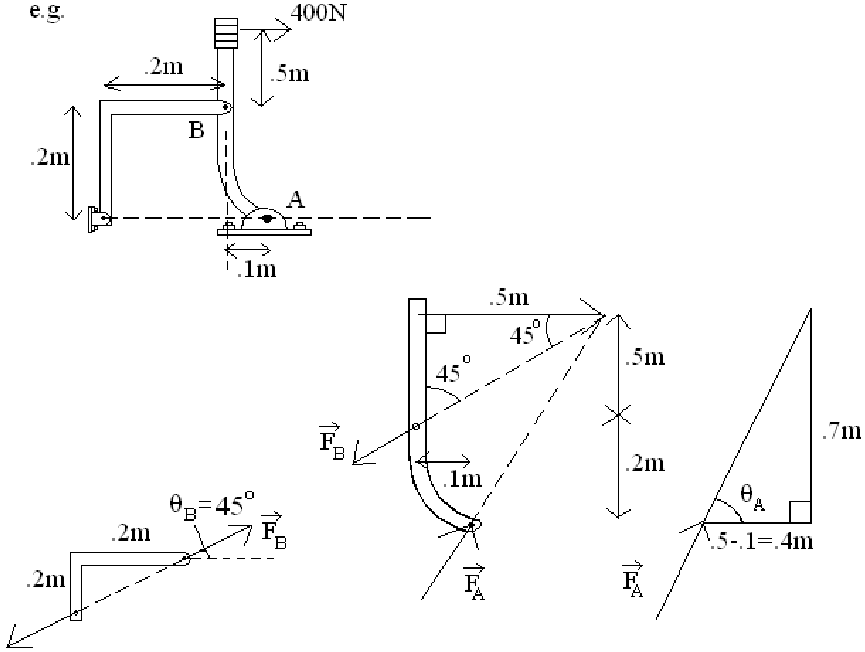
or parallel:



Knowing about two and three force members can sometimes simplify calculations and can also serve as a visual check for free-body diagrams which you think contain forces at only a few locations.

e.g.
Given: An external force resisted by a two-force link and a pin. (see pic)
Find: The magnitude of the support reactions F_A and F_B and their directions θ_A and θ_B.

e.g.



$$\theta_B = \tan^{-1}\frac{.2}{.2} = \boldsymbol{45°}$$

$$\theta_A = \tan^{-1}\frac{.7}{.4} = \boldsymbol{60.3°}$$

Looking at the free body diagram of the three force member:

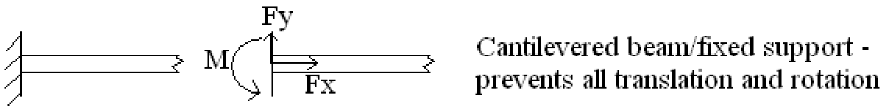
$$\sum Fx : F_A \cos 60.3° + 400 - F_B \cos 45° = 0$$

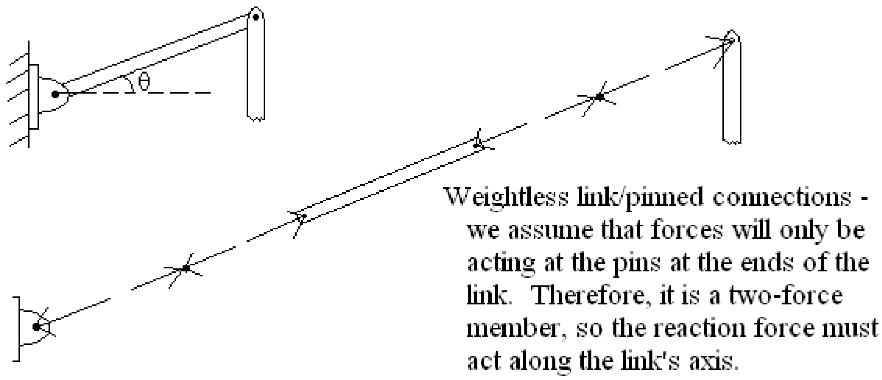
$$\sum Fy : F_A \sin 60.3° - F_B \cos 45° = 0$$

Solving $\rightarrow F_A$ = ***1.07kN*** $\quad F_B$ = ***1.32kN***

Trusses

Assumption: no friction, connections treated as points





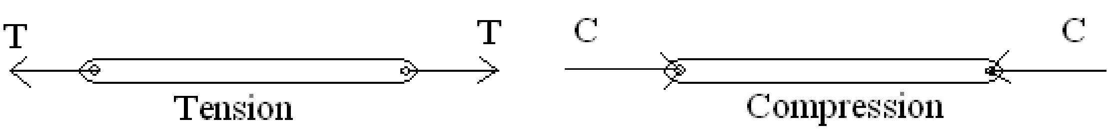
The weightless link will become important in the following section when we talk about trusses.

A truss is a structure composed of slender members joined together at their end points.

Truss assumptions:

1) All loadings are applied at the joints.
2) The members are joined together by smooth pins. In cases where bolted or welded joint connections are used, this assumption is satisfactory, provided the joining members intersect at a common point.

So, each truss member acts as a two-force member. Therefore, the forces at the ends of the member must be directed along the axis of the member.



The Method of Joints

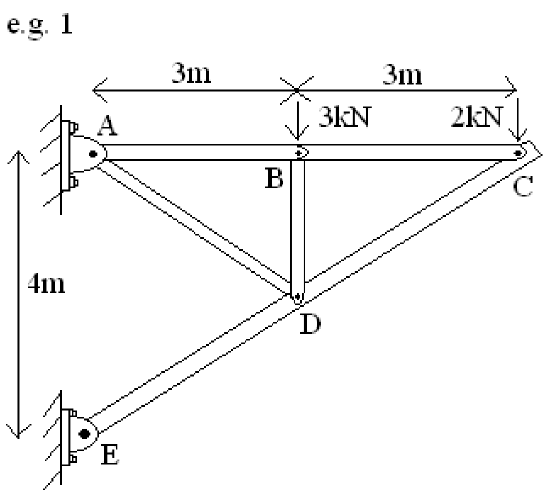
To find support reactions, we would normally ignore the forces within the members since they are internal to the support reaction's free body diagram. Instead, if we consider the equilibrium at each joint of the truss, then the member forces become external forces on the FBDs of the joints. The force system acting at each joint is concurrent. $\sum \text{Fx} = 0$ and $\sum \text{Fy} = 0$ must be satisfied for equilibrium. Therefore, we should start at a joint having at least one known and at most two unknown forces.

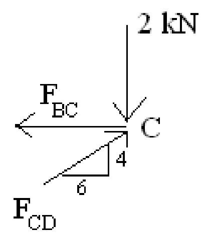
The correct sense of direction of an unknown member force (tensile or compressive) can in many cases be determined by "inspection." In more complicated cases, the sense of the unknown member force can be assumed. A positive answer indicates that you guessed correctly. A negative answer indicates that the sense shown on your FBD must be reversed. In this latter case, write the answer as positive but change (C) to (T) or vica-versa.

e.g. 1

Given: Dimensions of a truss for a balcony. Resultant loads act at B and C.

Find: The force in each member and state compression (C) or tension (T).

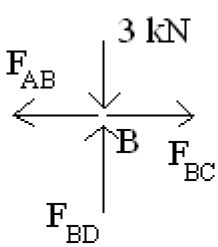




$$\xrightarrow{+} \sum Fx : F_{CD}\left(\frac{6}{\sqrt{6^2+4^2}}\right) - F_{BC} = 0$$

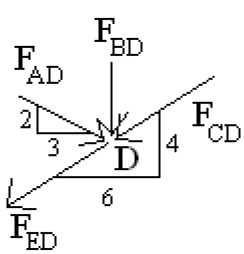
$$+\uparrow \sum Fy : F_{CD}\left(\frac{4}{\sqrt{6^2+4^2}}\right) - 2 = 0$$

F_{CD} = ***3.61kN (C)*** $\quad$ F_{BC} = ***3kN (T)***



$$\xrightarrow{+} \sum Fx : 3 - F_{AB} = 0 \qquad F_{AB} = \textbf{\textit{3kN (T)}}$$

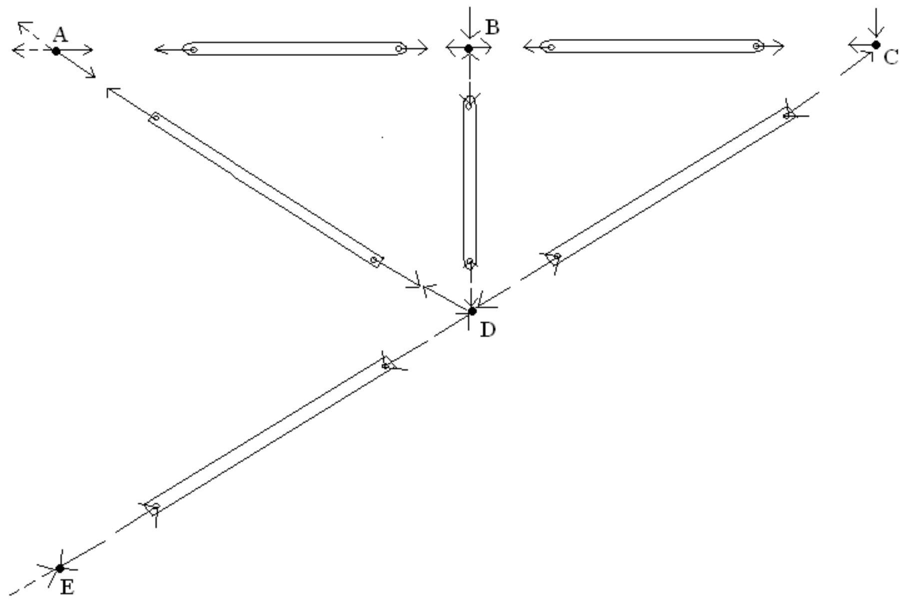
$$+\uparrow \sum Fy : F_{BD} - 3 = 0 \qquad F_{BD} = \textbf{\textit{3kN (C)}}$$



$$\xrightarrow{+} \sum Fx : F_{AD}\left(\frac{3}{\sqrt{3^2+2^2}}\right) - 3.61\left(\frac{6}{\sqrt{6^2+4^2}}\right) - F_{ED}\left(\frac{6}{\sqrt{6^2+4^2}}\right) = 0$$

$$+\uparrow \sum Fy: -3 - F_{AD}\left(\frac{2}{\sqrt{3^2+2^2}}\right) - 3.61\left(\frac{4}{\sqrt{6^2+4^2}}\right) - F_{ED}\left(\frac{4}{\sqrt{6^2+4^2}}\right) = 0$$

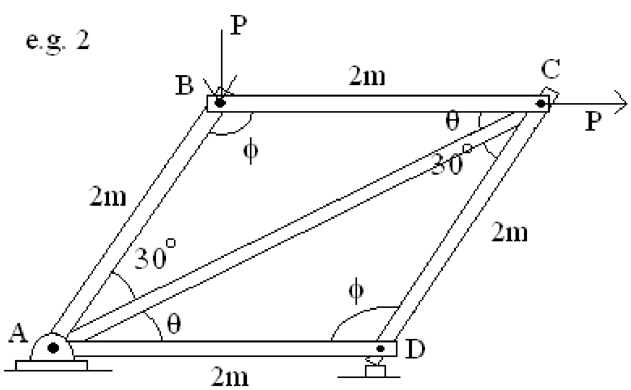
F_{AD} = -2.7 ⇒ F_{AD} = ***2.7kN (T)*** *F_{ED} = -6.31 ⇒ F_{ED} =* ***6.31kN (C)***

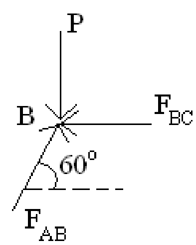


e.g. 2

Given: Dimensions of a truss (see pic below). T_{max} = 2kN, C_{max} = 1.2kN, for any member. Vertical external force P at B and horizontal force at C of equal magnitude P.

Find: Pmax





θ = 30° (law of sines)

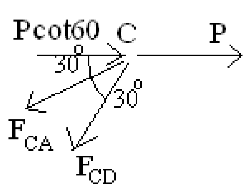
ϕ = 120°

$$\xrightarrow[+]{} \sum Fx: F_{AB}\cos 60° - F_{BC} = 0$$

$$+\uparrow \sum Fy: F_{AB}\sin 60° - P = 0$$

$$F_{AB} = \frac{P}{\sin 60°} = 1.155P \ \ (C)$$

$$F_{BC} = P\cot 60° = .577P \ \ (C)$$

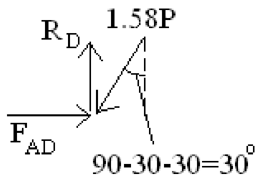


$$\xrightarrow[+]{} \sum Fx: P + P\cot 60° - F_{CA}\cos 30° - F_{CD}\cos 60° = 0$$

$$+\uparrow \sum Fy : -F_{CA} \sin 30° - F_{CD} \sin 60° = 0$$

$$F_{CA} = \frac{P + P \cot 60°}{\cos 30° - \sin 30° \cot 60°} = 2.73P \ \ (T)$$

$$F_{CD} = \frac{-2.73P \sin 30°}{\sin 60°} = -1.58P \Rightarrow F_{CD} = 1.58P \ \ (C)$$



$$\xrightarrow{+} \sum Fx : F_{AD} - 1.58P \sin 30° = 0$$

$$F_{AD} = 1.58 \sin 30° P = .79P \ \ (C)$$

Tension$_{max}$ = 2.73P
2kN = 2.73P

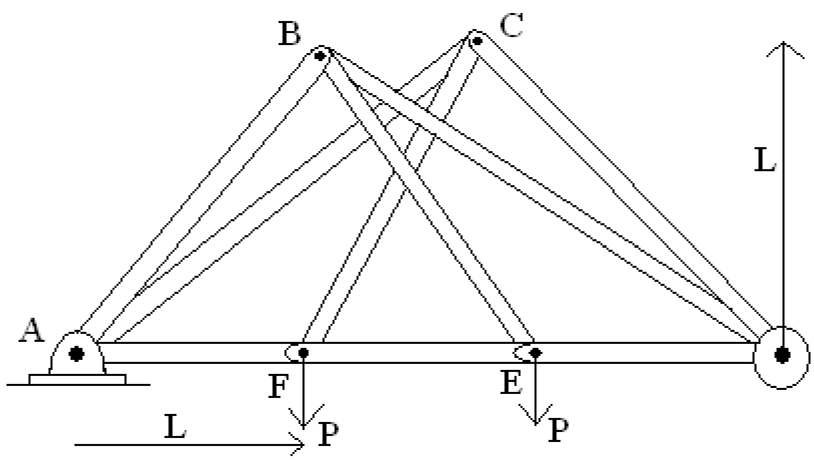
$$P = \frac{2}{2.73} = 732.6N$$

Compression$_{max}$ = 1.58P
1.2kN = 1.58P

$$P = \frac{1.2}{1.58} = 759.4N$$

P_{max} = 732.6N ≈ ***732N*** *(at that point, member CA would be pulled apart)*

e.g. 3
Given: Dimensions of a double scissors truss and resultant forces at joint E and F.
Find: Force in each member.



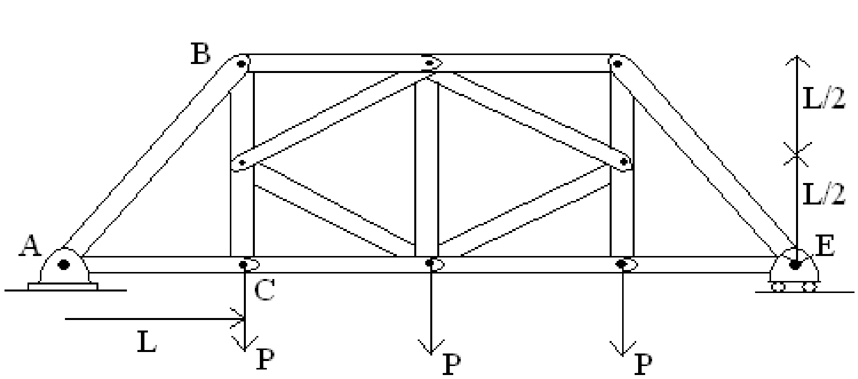
This problem is unusual because all joints have more than two unknowns, even if support reactions are known. However, the problem can be solved with the method of joints, by starting at joint E or F. $\sum F_{Ey} = 0$

$$\Rightarrow F_{EB} = \frac{P}{\sin 45°} \ \ (T)$$

Then, move on to joint B.

e.g. 4
Given: Dimensions of a K truss and resultant forces at H,G, and F shown below.
Find: Force in each member.

Here, we must calculate support reactions before we can do anything.



$R_A = R_Ay = R_E = \frac{3}{2}P$

Start at joint A or E;

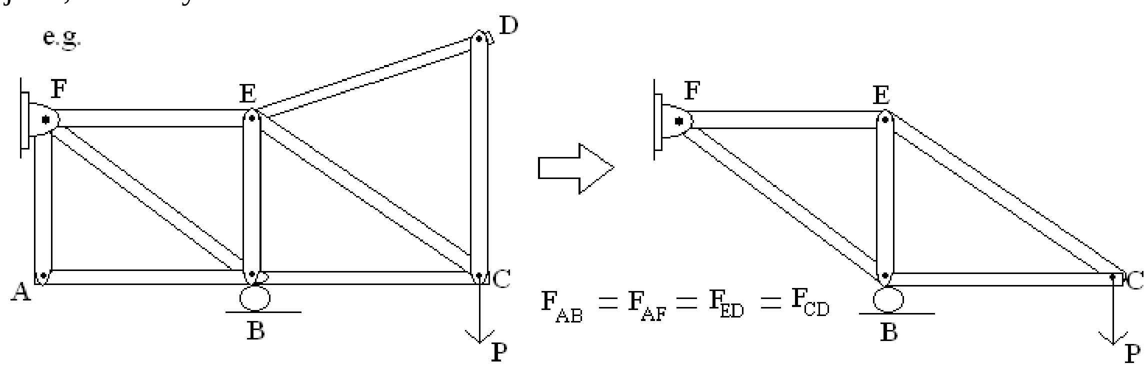
i.e., $\sum F_{Ay} = 0 \Rightarrow F_{AB} = \frac{3/2 * P}{\sin 45°}$ *(C)*

$\sum F_{Ax} = 0 \Rightarrow F_{AC} = 3/2 * P$ *(T)*

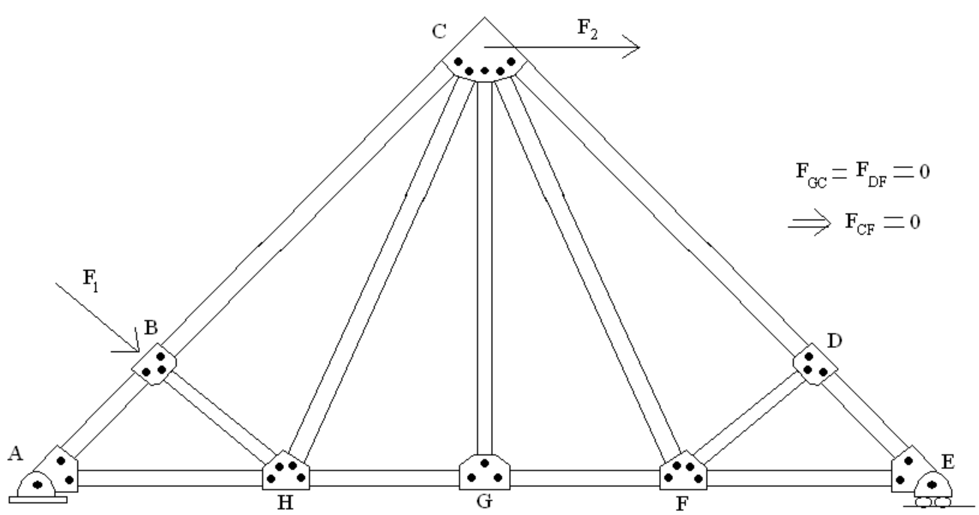
And continue from there to solve the problem.

<u>Zero Force Members</u>

If two elements meet at a joint and are *not* collinear and there is no external force at that joint, then they're both zero force members.

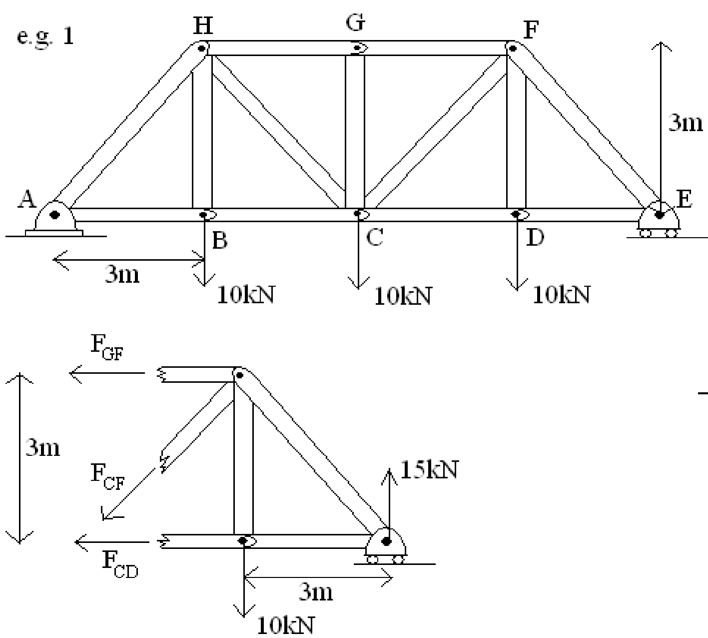


If three elements meet at a joint, two of which *are* collinear, and there is no external force at that joint, then the third member is a zero-force member.



<u>The Method of Sections</u>
The method of sections is usually used to find the loadings on just a few beams, efficiently. The basic idea is that if a body is in equilibrium, then any part of the body is also in equilibrium. Rather than dismembering every beam and applying particle equilibrium at each joint, we can "slice" through the whole structure and apply rigid body equilibrium to either of the two portions. $\sum Fx = 0, \sum Fy = 0, \sum Mo = 0$ must be satisfied for equilibrium. O can be anywhere in space. The other equations for equilibrium can also be used (see the beginning of the section "Equilibrium of a rigid body"). We should create a section with at least one known and at most three unknowns.

e.g. 1
Given: Bridge truss dimensions and equivalent forces at B, C, D.
Find: F_{GF}, F_{CF}, F_{CD}



Support reaction at E;

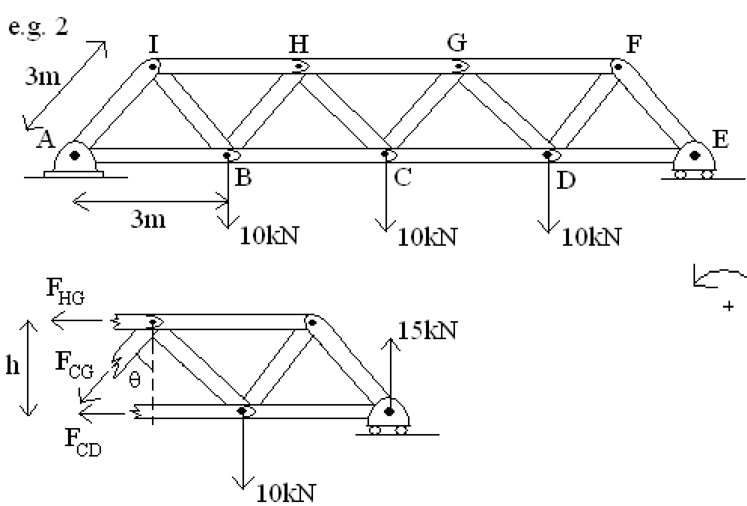
$$\sum M_A : R_E(12) - 10(3) - 10(6) - 10(9) = 0$$
$$\Rightarrow R_E = 15kN$$

Member forces;

$$\sum M_F : 15(3) - F_{CD}(3) = 0$$
$$+\uparrow \sum Fy : 15 - F_{CF}\cos 45° - 10 = 0$$
$$\xrightarrow{+} \sum Fx : -F_{GF} - 7.07\sin 45° - 15 = 0$$

F_{CD} = ***15kN (T)***
F_{CF} = ***7.07kN (T)***
F_{GF} = ***20kN (C)***

e.g. 2
Given: Warren truss dimensions and equivalent forces at B, C, D shown below.
Find: F_{GH}, F_{CG}, F_{CD}



Support reaction at E;
R_E *= 15kN (same as e.g. 1)*

$$\theta = \sin^{-1}\left(\frac{1.5}{3}\right) = 30°$$

$$h = \sqrt{3^2 - 1.5^2} = 2.60m$$

$$\sum M_G : 15(1.5+3) - 10(1.5)$$
$$- F_{CD}(2.60) = 0$$

$$+\uparrow \sum Fy : 15 - FCG\cos 30°$$
$$-10 = 0$$

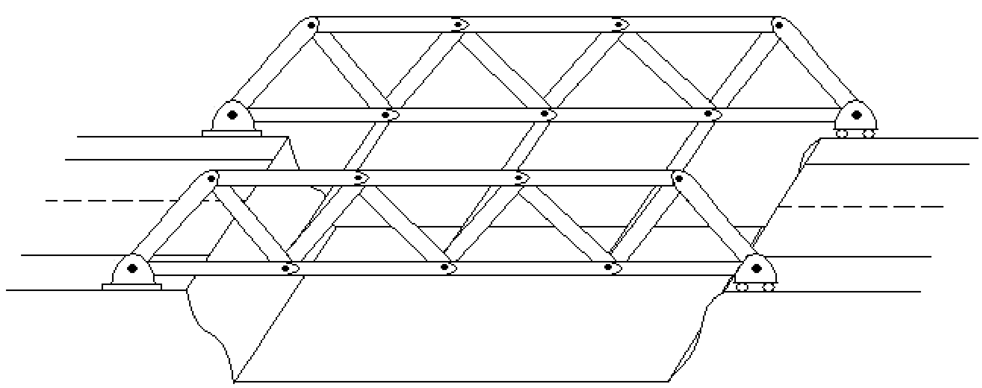
$\xrightarrow[+]{} \sum Fx : -F_{HG} - 5.77 \sin 30°$
$- 20.2 = 0$

F_{CD} = ***20.2kN (T)*** F_{CG} = ***5.77kN (T)*** F_{HG} = *-23.1* = ***23.1kN (C)***

Comparing e.g. 1 and e.g. 2, the bridge seems to be more economical, for this kind of loading.

note: The method of joints and the method of sections can be easily applied to 3D "space trusses." For the method of joints, there would be 3 equations for particle equilibrium at each joint. For the method of sections, there would be 6 equations $\sum \text{Fx} = \sum \text{Fy} = \sum \text{Fz} = 0, \sum \text{Mx} = \sum \text{My} = \sum \text{Mz} = 0$.

Remember that one of our truss assumptions was that the loads were applied only at the joints. Our truss models pictured in the previous examples show loads only at the joints. But, in real-life, is this true? The answer is yes, as we can see in the diagram below.



The load from the road and vehicles rests entirely on the beams, as we can see in the exaggerated diagram. The beams are connected to the bottom joints of the trusses, so the load is transferred entirely to those bottom joints. Our assumption is thus valid. Finally, the truss rests on the ground at each end of the valley, so the load is entirely transferred to the ground. The beams connecting the top joints of the trusses are not shown but typically would be present.

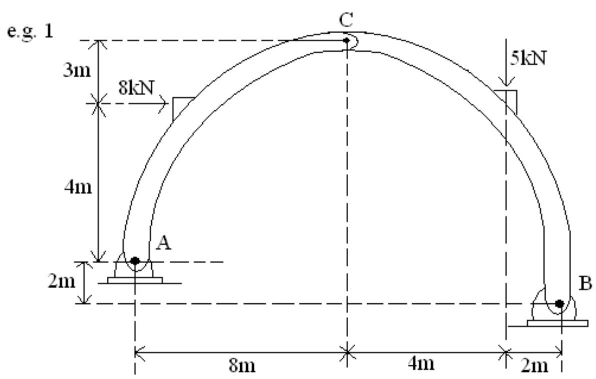
It's always important not only to know how to analyze a *component* of a structural system, such as the truss examples we have done in this section, but also to understand the *load path* for the entire system. The above diagram and explanation described the load path for a truss bridge. The load path for a truss roof in a gymnasium would be different. We will take a more quantitative look at the load paths for a simple building in a different outline.

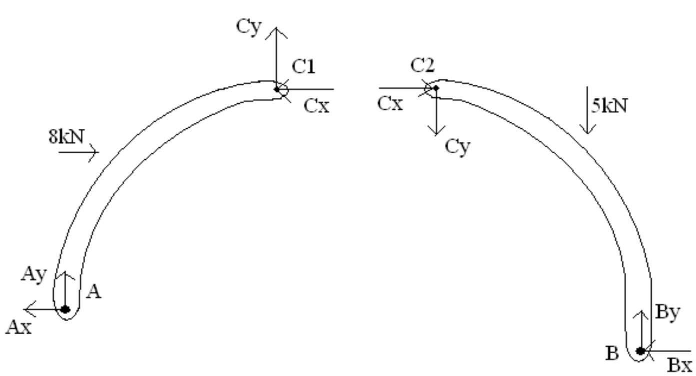
Equilibrium of a rigid body + Newton's 3rd Law

We've already analyzed frames and machines in the section "Equilibrium of a rigid body", but if we apply the same principle that was used for the "method of sections" – if a body is in equilibrium then any part of the body is also in equilibrium – then frames and machines (or any kind of structure) can be broken into pieces and analyzed using $\sum Fx = 0, \sum Fy = 0, \sum Mo = 0$ (or the other equations for rigid body equilibrium) for each piece.

Important: Forces common to any two contacting pieces must act with equal and magnitudes and opposite sense on the respective members (Newton's 3rd Law). These two principles – equilibrium for each part, and equal and opposite forces between parts – enables us to find support reactions when there are fewer "knowns" than in previous problems in this chapter, and now we can find internal reactions which we could not do at all before.

e.g. 1
Given: Three-hinged arc with external forces applied at locations shown.
Find: Reactions at A, B, C.





3 equations per free-body diagram
$\Rightarrow$ *6 total equations = 6 unknowns*

note: If there was an external force at C, it would go on either the right piece or the left, but NOT both

left side:

$$+\sum M_{C1} : -Ay(8) - Ax(4+3) + 8(3)$$
$$= 0$$

$$+\sum M_{A} : -8(4) + Cx(4+3) + Cy(8)$$
$$= 0$$

$$+\uparrow \sum Fy : Ay + Cy = 0$$

right side: $\circlearrowleft + \sum M_{C2} : By(4+2) - Bx(2+4+3) - 5(4) = 0$

$\circlearrowleft + \sum M_B : Cy(4+2) - Cx(2+4+3) + 5(2) = 0$

$+\uparrow \sum Fx : Cx - Bx = 0$

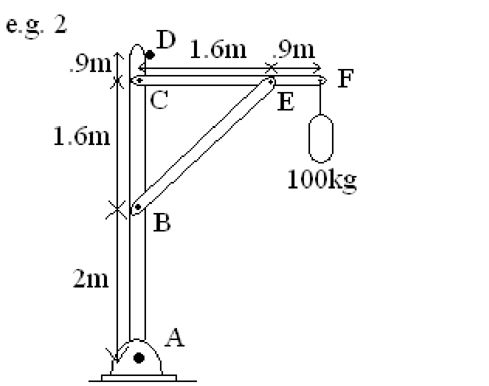
From $\sum M_B$, $Cy = \dfrac{Cx(2+4+3) - 5(2)}{4+2}$

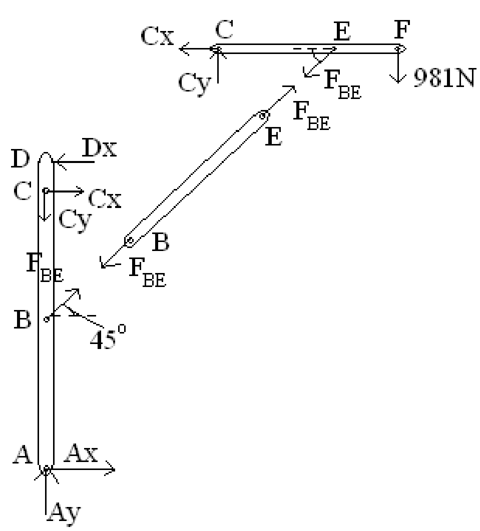
Substitute into $\sum M_A$, *Cx =* ***2.386kN*** *Cy =* ***1.91kN*** *Bx =* ***2.386kN*** *By =* ***6.91kN***

Ay = -1.91$\Rightarrow$*Ay =* ***1.91kN (down)*** *Ax =* ***5.6kN***

note: As the structures become more complex, it is easy to see that a calculator which can solve systems of linear and nonlinear equations simultaneously, can come in quite handy. The TI-89 calculator is one such calculator which has this capability.

e.g. 2
Given: Frame supported by a hinge at A and a roller at D. 100kg mass at F.
Find: Support reactions and all internal reactions at labeled joints. (see pic below)





entire frame:

$\circlearrowleft + \sum M_A : -981(2.5) + Dx(4.5) = 0 \Rightarrow Dx =$ ***545N***

$\xrightarrow{+} \sum Fx : Ax - Dx = 0 \Rightarrow Ax =$ ***545N***

$+\uparrow \sum Fy : Ay - 981 = 0 \Rightarrow Ay =$ ***981N***

horizontal member:

$\circlearrowleft + \sum M_C : -981(2.5) - F_{BE} \sin 45°(1.6) = 0$

$\xrightarrow{+} \sum Fx : -Cx - F_{BE} \cos 45° = 0$

$+\uparrow \sum Fy : Cy - 981 - F_{BE} \sin 45° = 0$

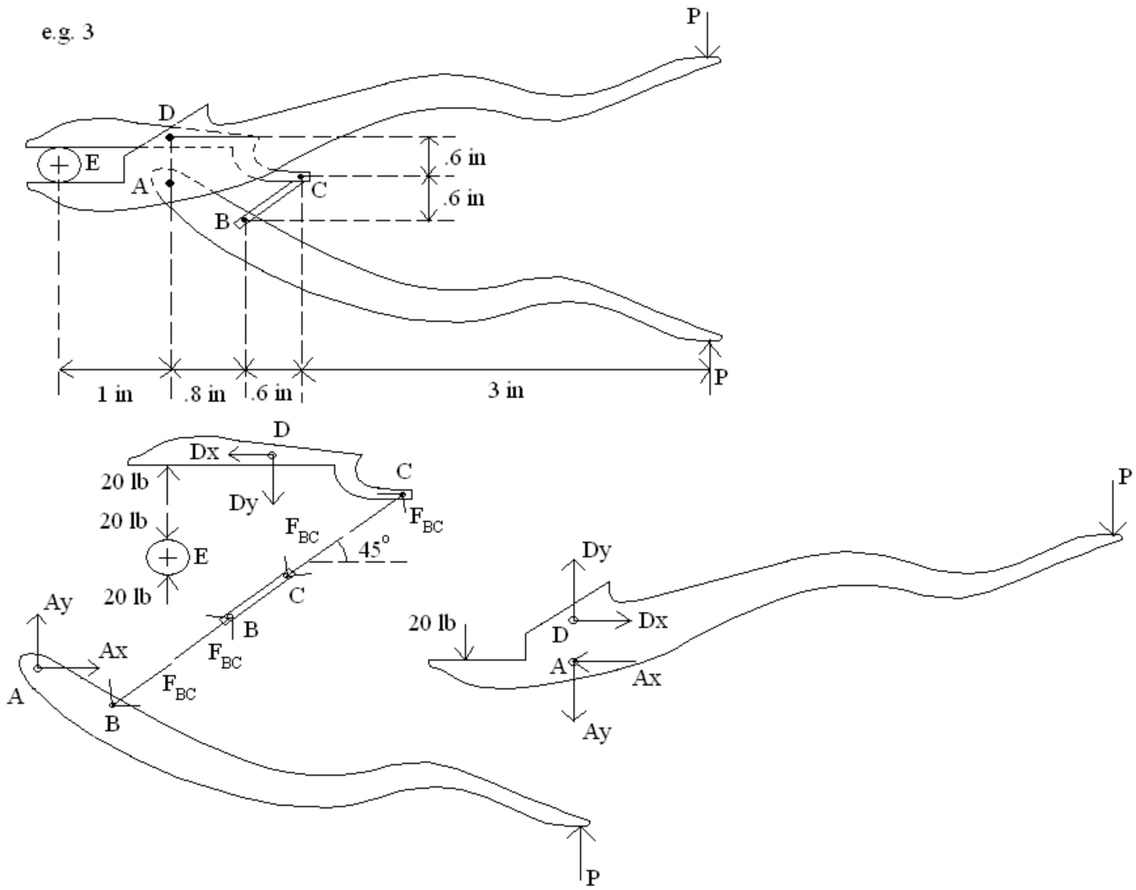
F_{BE} *= -2168*$\Rightarrow$$F_{BE}$ = ***2168N (C)***

Cx = ***1533N*** *Cy = -552*$\Rightarrow$*Cy =* ***552N (down)***

e.g. 3

Given: Dimensions of pruning shears. We need a cutting (normal/contact) force of 20 lb.

Find: Squeezing force P applied at location shown below. (see pic below)



upper blade:

$$\sum M_D : F_{BC}\sin 45°(1.4) + F_{BC}\cos 45°(.6) - 20(1) = 0 \Rightarrow F_{BC} = \mathit{14.14\ lb}$$

$$\xrightarrow{+} \sum Fx : F_{BC}\cos 45° - Dx = 0 \Rightarrow Dx = \mathit{10.0\ lb}$$

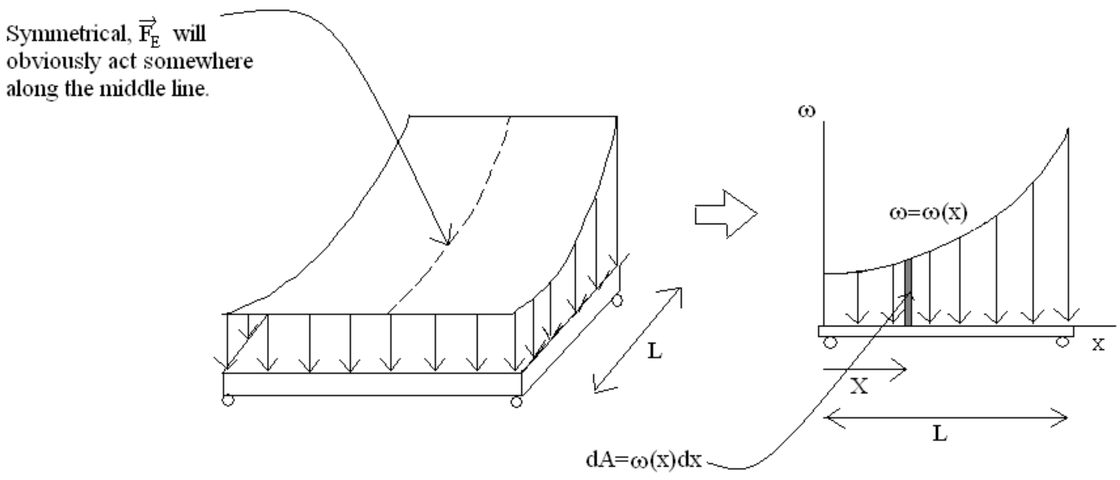
lower jaw/upper arm:

$$\sum M_A : 20(1) - 10(.6) - P(3 + .6 + .8) = 0 \Rightarrow P = \boldsymbol{3.2\ lb}$$

note: started with a piece that had at least one known and at most three unknowns.

note: these pruning shears enable one to multiply their grip force by over 5 times.

Distributed loading



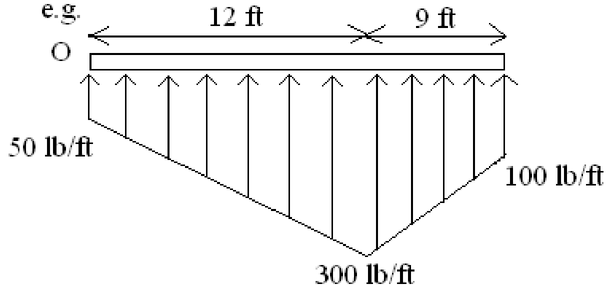
$F_R = A = \int_L \omega(x)dx \qquad M_{RO} = \int_L x\omega(x)dx$

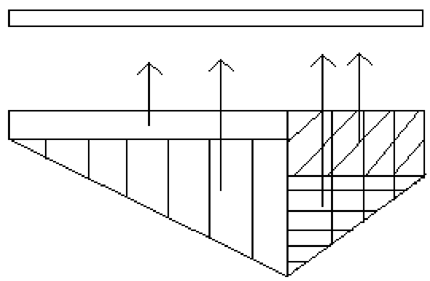
$$\text{“d”} = \bar{x} = \frac{\int_L x\omega(x)dx}{\int_L \omega(x)dx}$$

$\omega(x) =$ density function = force per unit length (i.e. $40(x^2+2)$ could be $\omega(x)$ for the ramp above). For uniform density, the magnitude of the constant (i.e. $\omega(x) = 40$) affects F_R but not $\bar{x}$.

e.g.
Given: Density at the three vertices of the triangular + rectangular distribution.
Find: F_R and its location measured from O, for equivalency.
(see pic below)

e.g.





$$\sum F = lw + \frac{1}{2}BH + lw + \frac{1}{2}BH = 12(50) + \frac{1}{2}(300-50)(12) + 9(100) + \frac{1}{2}(300-100)(9)$$

$$= \mathit{3900lb} = \boldsymbol{F_R}$$

$$\sum M = 12(50)(6) + \frac{1}{2}(300-50)(12)(8) + 9(100)(12+4.5) + \frac{1}{2}(300-100)(9)(12+3) = 43950lb * ft$$

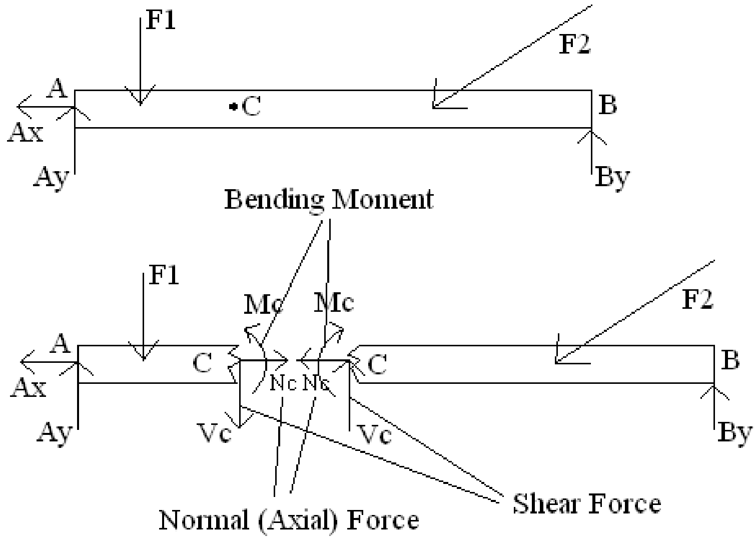
$$= \boldsymbol{M_o}$$

$$\bar{x} = \frac{43950}{3900} = \boldsymbol{11.27\ ft}$$

note: For a triangular distribution, the equivalent force $\frac{1}{2}\text{BH}$ is located a distance along its base (from the peak side) equal to $\frac{1}{3}\text{length}$.

Beams – bending moment diagrams and shear force diagrams

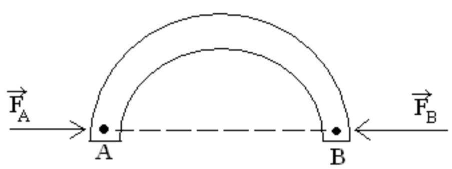
External forces on a frame create "internal" forces at each connection, and they also create internal forces within each member, which tend to deform it.



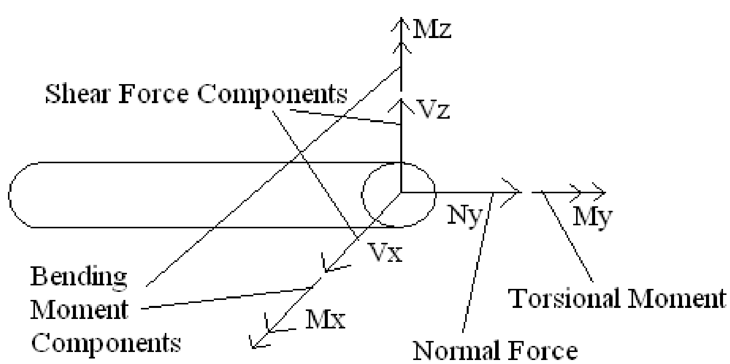
Nc is an axial (tensile or compressive) force, which is a type of force we've already dealt with. Note equal and opposite forces on each piece of the beam, from Newton's 3rd Law.

Shear and bending moments within a beam are non-zero if:

1) The beam is a two force member but is not straight.



2) The beam is subjected to multiple forces (other than at the joints).



note: The sense of V and M drawn to the left and above is treated as positive in structural engineering.

<u>Cutting Method</u>

We will now develop methods for drawing shear force and bending moment diagrams that illustrate shear and moment magnitudes at every point along a beam. We will not be concerned with axial forces, since all external forces will act perpendicular to the members.

Picking the right size beam requires knowledge of V and M at each point along the beam's axis. As we will see in the next outline "Mechanics of Materials" and the following outlines on concrete and steel design, one can then use appropriate formulas to determine the required cross sectional area. Graphs of V and M as functions of x are called shear force diagrams (SFD) and bending moment diagrams (BMD).

The cutting method is useful for: *simple* distributed loads where a complicated w(x) function is not given.

Process:

1) Find support reactions
2) Find equations for V and M from equilibrium equations
3) Find peak values

note: signs are important!

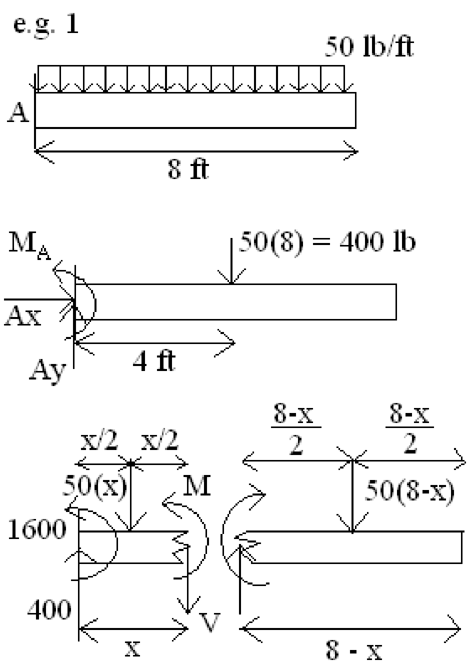
note: assumed sense of V and M is important!

Look over the following examples to fully understand this method.

e.g. 1

Given: 8 ft beam connected with fixed support and subjected to distributed load shown.

Find: SFD and BMD for the beam.



Support reactions:

Ax = 0

$$+\uparrow \sum Fy : Ay - 400 = 0 \Rightarrow Ay = 400lb$$

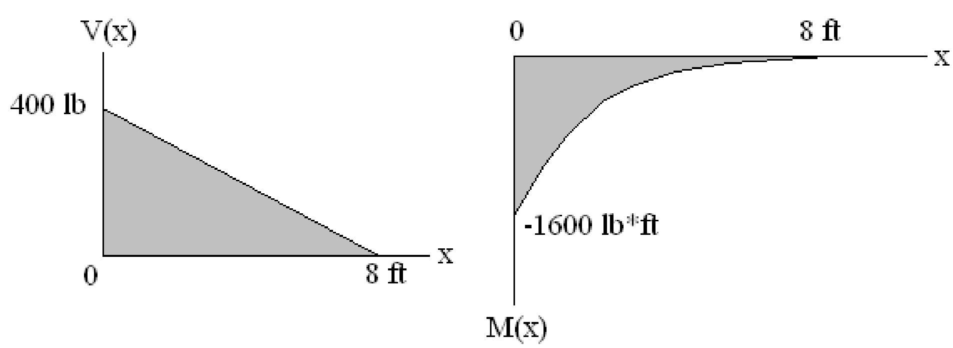
$$+\circlearrowleft \sum M_A : M_A - 400(4) = 0 \Rightarrow M_A = 1600lb * ft$$

magnitude and location of distributed load on each piece must be in terms of x as shown.

$$+\uparrow \sum Fy : 400 - 50(x) - V = 0 \Rightarrow V = 400 - 50(x)$$

$$+\circlearrowleft \sum M_{cut} : 1600 - 400(x) + 50(x)(\frac{x}{2}) + M = 0 \Rightarrow$$

$$M = 400x - 25x^2 - 1600$$

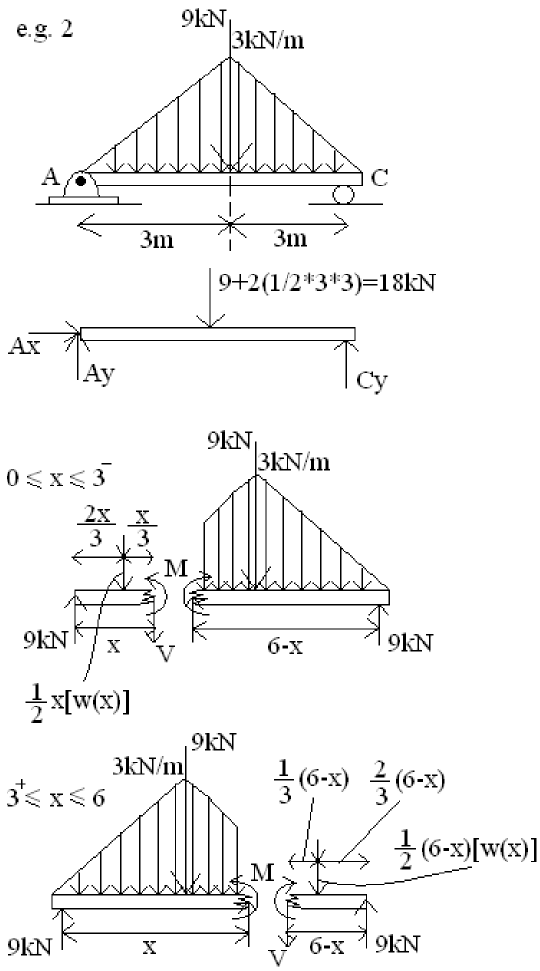


note: Fixed beams like this are sometimes tapered (thicker near the support) in order to be cost effective, since we can see that the forces are large only near the support.

e.g. 2

Given: Beam supported at both ends. Distributed load and concentrated force (see pic below).

Find: SFD and BMD for the beam.



Support reactions:
Ax = 0
Ay = Cy = 9kN (symmetry)

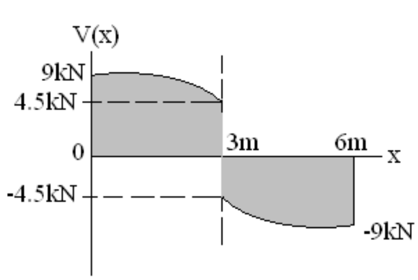
$0 \leq x \leq 3^-$

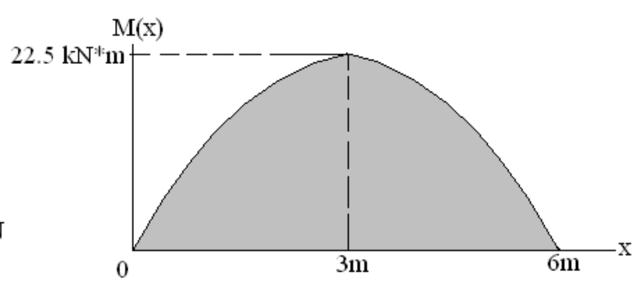
choose left side: from similar triangles (or y = mx + b), $\frac{w(x)}{x} = \frac{3kN/m}{3m} \Rightarrow w(x) = 1\frac{kN}{m^2}(x)$

$$\uparrow \sum Fy : 9 - \frac{1}{2}x^2 - V = 0 \Rightarrow V = 9 - \frac{1}{2}x^2$$

$$\circlearrowleft + \sum M_{cut} : 9(6-x) - \frac{1}{2}(6-x)^2[\frac{1}{3}(6-x)] - M = 0$$

$$\Rightarrow M = 9(6-x) - \frac{1}{6}(6-x)^3$$





<u>Graphical Method</u>

This method is useful for: Anytime, but especially for complicated distributed loads where w(x) is give as a function

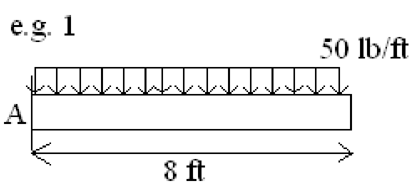
Process:
1) Find support reactions
2) Find equations for V and M using integration
3) Find peak values

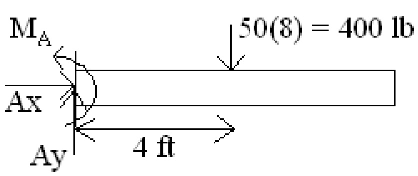
$V(x) = -\int w(x)dx$ $\qquad$ $M(x) = \int V(x)dx$

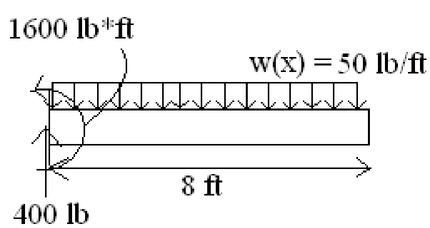
note: A point load changes V(x) AT THAT POINT by the amount of the point load. An external moment changes M(x) AT THAT POINT by the amount of the external moment. Work from left to right! Assumed sense of V and M is still important, and w(x) is positive in downward direction.

Look over the following examples to fully understand this method.

e.g. 1
Given: 8 ft beam connected with fixed support and subjected to distributed load shown.
Find: SFD and BMD for the beam.







Support reactions:

Ax = 0

$$+\uparrow \sum Fy : Ay - 400 = 0 \Rightarrow Ay = 400lb$$

$$+\circlearrowleft \sum M_A : M_A - 400(4) = 0 \Rightarrow M_A = 1600lb * ft$$

$$x = 0^- : V(0^-) = 0 \quad M(0^-) = 0$$

$$x = 0^+ : V(0^+) = V(0^-) + 400 = 400$$

$$M(0^+) = M(0^-) - 1600 = -1600$$

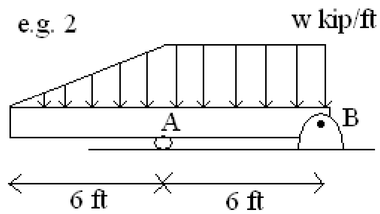
$$0^+ \le x \le 8 : V(x) = V(0^+) - \int_0^x 50dx = 400 - 50x$$

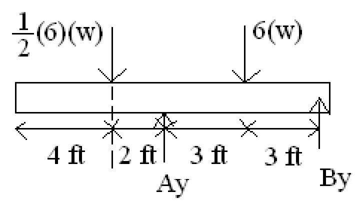
$$M(x) = M(0^+) + \int_0^x 400 - 50xdx$$

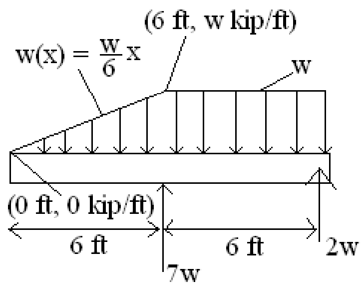
$$= -1600 + 400x - 25x^2$$

(compare with e.g.1 for "cutting method")

e.g. 2
*Given: Beam shown will fail for M>30 kip*ft or V>8kip at any point.*
Find: Largest distributed load w possible (see pic below).







Support reactions:

$$\circlearrowleft + \sum M_B : 6(w)(3) + \frac{1}{2}(6)(w)(8) - Ay(6) = 0 \Rightarrow Ay = 7w$$

$$\uparrow \sum Fy : 7w - \frac{1}{2}(6)(w) - 6(w) + By = 0 \Rightarrow By = 2w$$

$$x = 0^- : V(0^-) = 0 \quad M(0^-) = 0$$

$$x = 0^+ : V(0^+) = V(0^-) + 0 = 0 \quad M(0^+) = 0$$

$$0^+ \le x \le 6^- : V(x) = V(0^+) - \int_0^x \frac{w}{6} x dx = -\frac{wx^2}{12}$$

$$M(x) = M(0^+) + \int_0^x -\frac{wx^2}{12} dx = -\frac{wx^3}{36}$$

$$x = 6^- : V(6^-) = -\frac{w(6)^2}{12} = -3w \quad M(6^-) = -\frac{w(6)^3}{36} = -6w$$

$$x = 6^+ : V(6^+) = V(6^-) + Ay = -3w + 7w = 4w$$

$$M(6^+) = M(6^-) + 0 = -6w$$

$$6^+ \le x \le 12^- : V(x) = V(6^+) - \int_6^x w dx = 4w - w(x-6) = 10x - wx$$

$$M(x) = M(6^+) + \int_6^x 10w - wx dx = -6w + 10wx - \frac{wx^2}{2} - [10w(6) - \frac{w(6)^2}{2}]$$

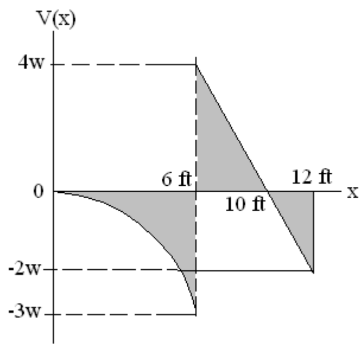
$$= -\frac{wx^2}{2} + 10wx - 48w$$

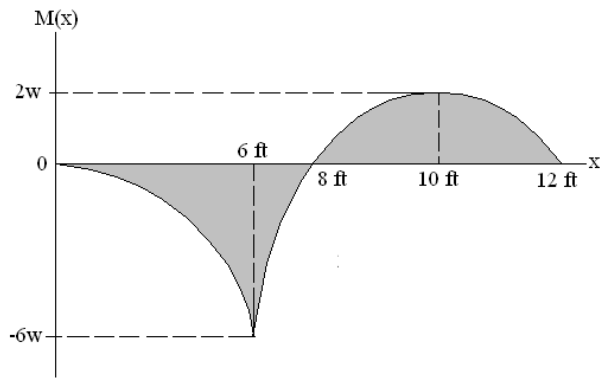
$$x = 12^- : V(12^-) = 10w - w(12) = -2w$$

$$M(12^-) = -\frac{w(12)^2}{2} + 10w(12) - 48w = 0$$

$$x = 12^+ : V(12^+) = V(12^-) + By = -2w + 2w = 0 \qquad OK$$

$$M(12^+) = M(12^-) + 0 = 0 \qquad OK$$



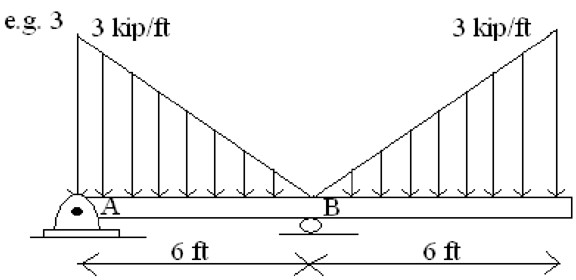


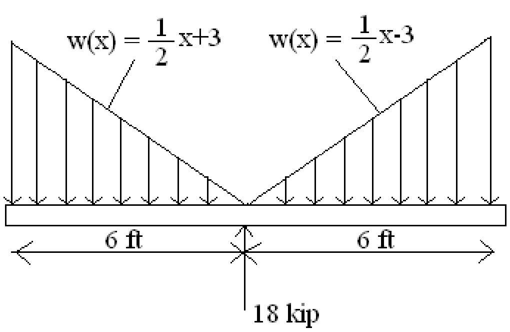
*M_{max} = 30 kip*ft = 6w*
$\Rightarrow$ *w_{max} = 5 kip/ft*

Vmax = 8 kip = 4w
$\Rightarrow$ *w_{max} = 2 kip/ft*

*w_{max} = **2 kip/ft***

e.g. 3
Given: Beam with supports shown. Symmetrical distributed loads.
Find: SFD and BMD for the beam.





Support reactions:
From symmetry, Ay = Ax = 0

By = $2*[\frac{1}{2}(6)(3)] = 18kip$

$$x = 0 : V(0) = 0 \quad M(0) = 0$$

$$0 \le x \le 6^- : V(x) = V(0) - \int_0^x -\frac{1}{2}x + 3dx = \frac{1}{4}x^2 - 3x$$

$$M(x) = M(0) + \int_0^x \frac{1}{4}x^2 - 3xdx = \frac{1}{12}x^3 - \frac{3}{2}x^2$$

$$x = 6^- : V(6^-) = \frac{1}{4}(6)^2 - 3(6) = -9$$

$$M(6^-) = \frac{1}{12}(6)^3 - \frac{3}{2}(6)^2 = -36$$

$$x = 6^+ : V(6^+) = V(6^-) + 18 = 9$$

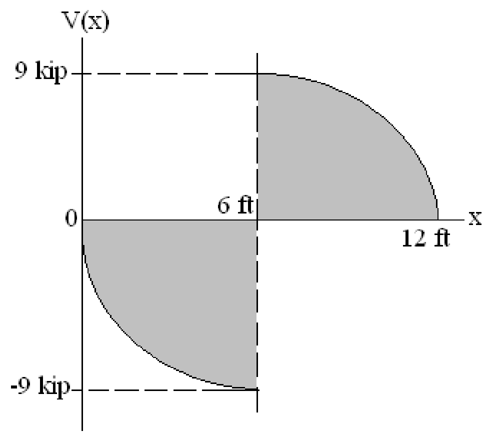
$$M(6^+) = M(6^-) + 0 = -36$$

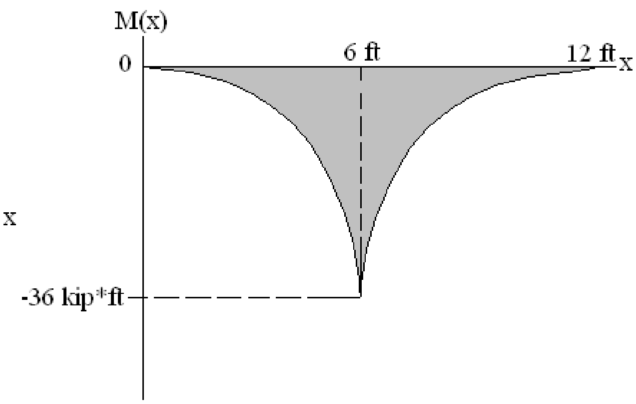
$$6^+ \le x \le 12 : V(x) = V(6^+) - \int_6^x \frac{1}{2}x - 3dx = 9 - \frac{1}{4}x^2 + 3x + 9 - 18 = 3x - \frac{1}{4}x^2$$

$$M(x) = M(6^+) + \int_6^x 3x - \frac{1}{4}x^2 dx = -36 + \frac{3x^2}{2} - \frac{1}{12}x^3 - 54 + 18$$

$$= -\frac{1}{12}x^3 + \frac{3}{2}x^2 - 72$$

$$x = 12: V(12) = 3(12) - \frac{1}{4}(12)^2 = 0 \quad OK \qquad M(12) = -\frac{1}{12}(12)^3 + \frac{3}{2}(12)^2 - 72 = 0 \quad OK$$

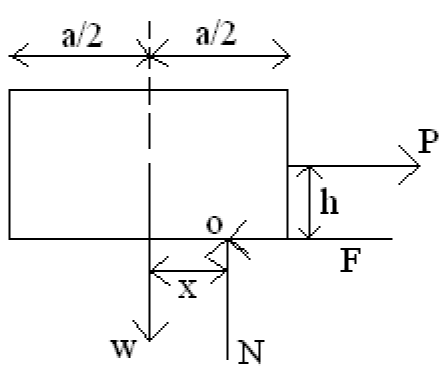




note: w(x) from $6^+ \le x \le 12$ is $\frac{1}{2}x - 3$ NOT just $\frac{1}{2}x$ because our origin is at the far left side of the beam, not the middle of the beam.

Friction

<u>Single contact block</u>



Moment equilibrium about point O is satisfied if Wx = Ph or $x = \frac{Ph}{W}$. The block will be on the verge of tipping if N acts at the right corner of the block.

Verge of tipping $\Rightarrow x = \frac{a}{2}$ check: $F \le \mu_s * N$

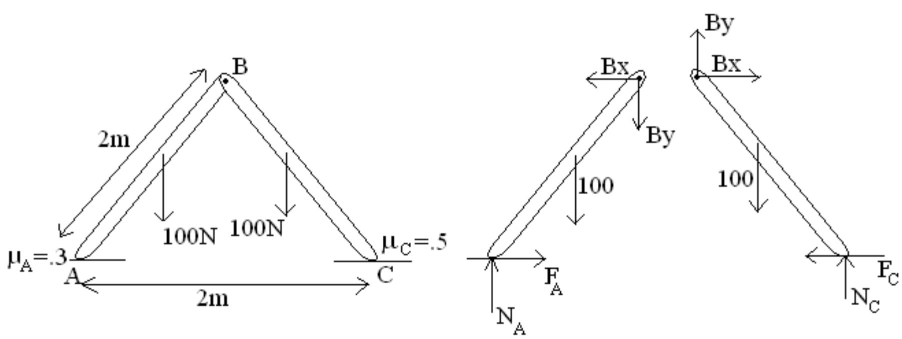
Equilibrium of the block also requires F = P

Verge of slipping $\Rightarrow F = \mu_s * N$ check: $x \le \frac{a}{2}$

Multiple contact points

Four possibilities:

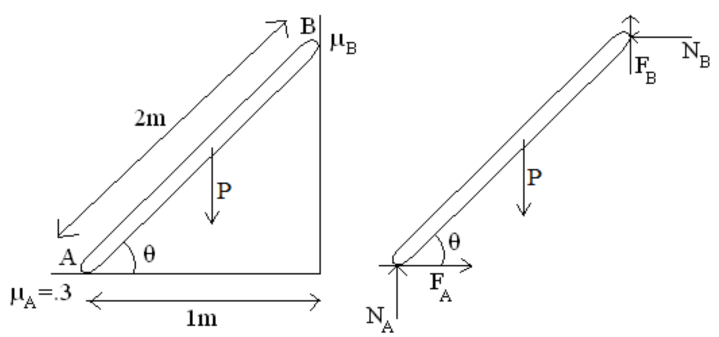
1) No unknown external forces, *no* contact points necessarily have impending motion.



6 unknowns
6 equations (rigid body equilibrium)

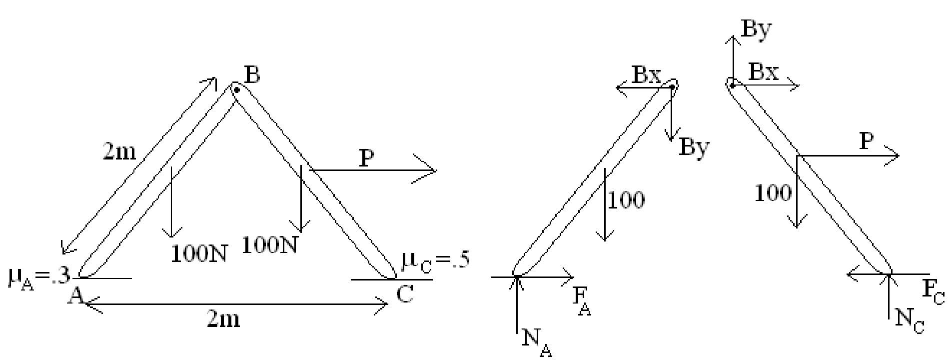
check: $F_A \leq \mu_A * N_A$
$F_C \leq \mu_C * N_C$

2) Maximizing an external force (minimizing an angle); *all* contact points on the verge of slipping.



5 unknowns
3 equations (rigid body equilibrium) + 2 equations ($F_A = \mu_A N_A$, $F_B = \mu_B N_B$)

3) Maximizing an external force, *some* (usually one) contacts points have impending motion (slipping or tipping)



7 unknowns
6 equations (rigid body equilibrium) + guess
- $F_A = \mu_A N_A$, if A slides first

check: $N_A \geq 0$
$F_C \leq \mu_C * N_C$

OR

- $F_C = \mu_C N_C$, if C slides first

check: $N_A \geq 0$

$F_A \leq \mu_A * N_A$

OR

- $N_A = 0$, if rotation about C occurs first.

check: $F_A = 0$

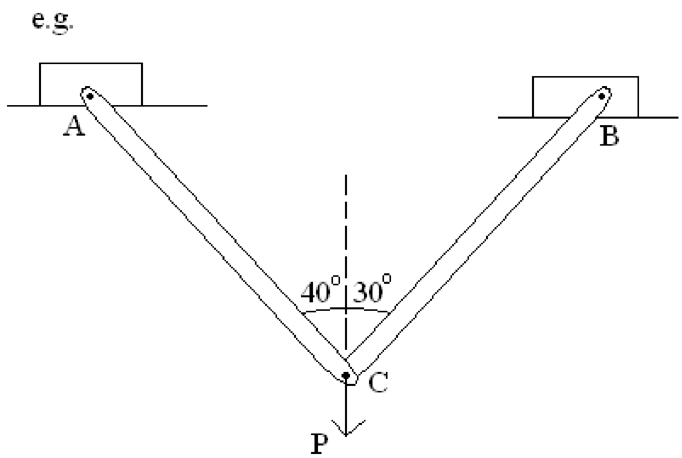
$F_C \leq \mu_C * N_C$

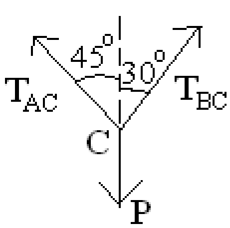
4) Maximizing an external force; forces contribute solely to tipping.
(case 4 problems are fairly common – an example would be finding the maximum load a crane truck can hold before tipping)

e.g. (case 3)

Given: Vertical force P applied at connection. Mass of each block = 6 kg. $\mu_A = .2$ $\mu_B = .8$.

Find: Largest force P before motion.





$$\xrightarrow{+} \sum Fx : T_{BC} \sin 30° - T_{AC} \sin 45° = 0$$

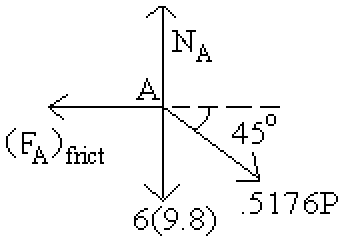
$$+\uparrow \sum Fy : T_{BC} \cos 30° + T_{AC} \cos 45° - P = 0$$

From $\sum Fx,\ T_{AC} = \dfrac{T_{BC} \sin 30°}{\sin 45°}$

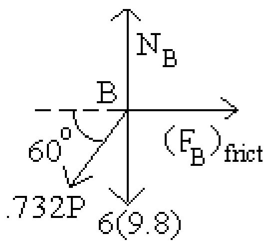
Substitute into $\sum Fy,\ T_{BC} \cos 30° + (\dfrac{T_{BC} \sin 30°}{\sin 45°}) \cos 45° - P = 0$

$$T_{BC} = \frac{1}{\cos 30° + \sin 30° \cot 45°} P \approx .732P$$

$$T_{AC} = \frac{\sin 30°}{\sin 45°(\cos 30° + \sin 30° \cot 45°)} P \approx .5176P$$



$$+\uparrow \sum Fy : N_A - 6(9.8) + .5176P\sin 45^\circ = 0$$
$$\Rightarrow N_A = 58.86 + .366P$$
$$\xrightarrow{+} \sum Fx : .5176P\cos 45^\circ - (F_A)\,frict = 0 \Rightarrow (F_A)\,frict = .366P$$



$$+\uparrow \sum Fy : N_B - 6(9.8) + .732P\sin 60^\circ = 0 \Rightarrow N_B = 58.86 + .634P$$
$$\xrightarrow{+} \sum Fx : .(F_B)\,frict - .732P\cos 60^\circ = 0 \Rightarrow (F_B)\,frict = .366P$$

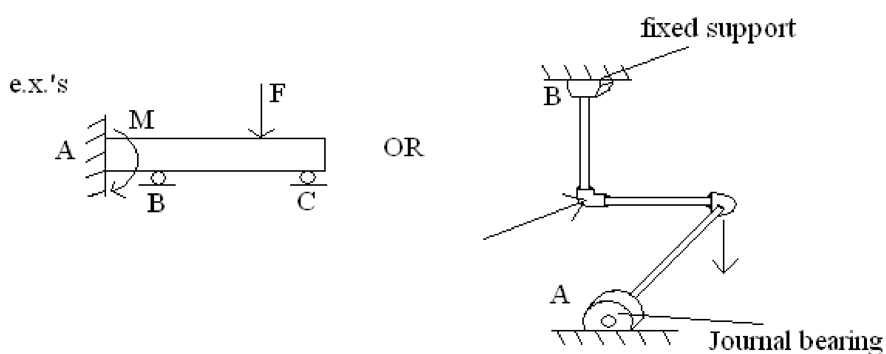
Guess: $(F_A)_{frict} = \mu_A N_A$
$.366P = .2(58.86+.366P) \Rightarrow P_{max} = \mathbf{40.2N}$

Check: $(F_B)_{frict} = .366(40.2) = 14.7N$
$\mu_B N_B = .8[58.86+.634(40.2)] = 67.5N$
$(F_B)_{frict} = 14.7 \leq \mu_B N_B = 67.5$ *OK*

note: could have broken apart the frame, applied rigid body equilibrium, and made our guess at the beginning or the end, in that manner. This would have been more difficult though since member lengths were not given.

A note on redundancy

Redundant supports are extra supports that are not necessary to hold the body in equilibrium.



Redundancy is generally good, but it creates too many unknowns for our 3 (2D) equations or 6 (3D) equations of static equilibrium. Note how most of the examples

throughout this chapter have used primarily pins or rollers, because multiple fixed supports $\Rightarrow$ redundancy.
The additional equations needed for redundant supports involve the physical properties of the body, which are studied in subjects dealing with the mechanics of deformation. The following chapters on Mechanics of Materials and Classical Structural Analysis deal with redundant systems. Redundant systems are commonly called "statically indeterminate" systems.

Works Cited

Hibbeler, R.C. Engineering Mechanics: Statics Tenth Edition. Pearson. Upper Saddle River, NJ 2004.

Johnson, Erik. Lecturer. Univ. of Southern California. CE205. Fall 2004.