# A Basic Overview of the Slope-Deflection Method 

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#### Abstract

In the early 1900s, civil engineers developed the Slope-Deflection Method as quick and relatively painless way of calculating the moments within structures that could not be solved using the three equations of equilibrium. Simply, equilibrium equations, which are used in the analysis, are expressed in terms of unknown joint displacements. An expression is used to relate the moment at each end of a member both to the end displacements. Each member will receive an equation to relate its rotations. then we will use a system of linear equations to relate the rotations. We can use this to find all of the unknown internal moments in indeterminate structures. With this rough idea, we can now attempt to understand how the slope deflection method works.


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## 1 Derivation of the Slope-Deflection Method

First, we will start the derivation by examining two different figures. Figure 1 is a uniform distributed load across a beam with some fixed end moments. Below that is the respective moment diagram. Figure 2 is the deformations of the member exaggerated on the vertical scale.


Fig. 1 Loading and Bending Moment Diagram for Example Beam.


Fig. 2 Deformations of Member AB Plotted to an Exaggerated Vertical Scale.

Notice, in Figure 2, that the deflection is related to the rotation on the ends and the internal moments present within the beam. The total amount of rotation can actually be expressed as a function of the deflection and rotation of the member. To do this, one must use some fundamentals from the Moment Area Method. Notice the tangent line at A and the distance that it makes from the tangent line to the support at B. This creates a relationship that can be used to relate the angle at A to the member. To relate the total rotation at point A, which we will
call $\varphi$, to the of tangent line A of the opposite side $t_{B A}$ the equation follows.

$$
\begin{equation*}
\varphi_{A}=\frac{t_{B A}}{L} \tag{1}
\end{equation*}
$$

Likewise, this is true on the other side.Therefore...

$$
\begin{equation*}
\varphi_{B}=\frac{t_{A B}}{L} \tag{2}
\end{equation*}
$$

Notice how these angles correspond to the opposite side tangential lines difference. Referencing Figure 2 again, we notice that $\varphi_{A}$ can be rewritten as $\varphi_{A}=\theta_{A}-\psi_{A B}$. Also, $\varphi_{B}$ can be rewritten as $\varphi_{B}=\theta_{B}-\psi_{B A}$. This simply transforms the total rotation angle into the sum of its parts. $\theta$ and $\psi$ correlate to the rotation of the member due to its fixed end moments and the rotation of the member due to the deflection of the member respectively. Substituting what we know into (1) and (2), we yield two new equations.

$$
\begin{align*}
\theta_{A}-\psi_{A} & =\frac{t_{B A}}{L}  \tag{3}\\
\theta_{B}-\psi_{B} & =\frac{t_{A B}}{L} \tag{4}
\end{align*}
$$

To express $t_{A B}$ and $t_{B A}$ in terms of the applied moments in figure 1 , we divide the moment curves in figure 1 by EI to produce M/EI curves and then take the moment underneath the curve at A and B. Notice in Figure (1) how the moment diagram only has two adding factors in the moment equation: a triangle and the moment due to the loading. The moment due to the loading we will call the fixed end moment. Also, notice how $M_{B A}$ has a larger influence on $t_{A B}$. This is because of the equation originating from the Moment-Area Method. To relate the distance from the tangent lines to the support to their respective moments and deflections we use the equations below.

$$
\begin{align*}
& t_{B A}=\left[\frac{M_{B A}}{E I} * \frac{L}{2} * \frac{2 L}{3}\right]-\left[\frac{M_{A B}}{E I} * \frac{L}{2} * \frac{L}{3}\right]-F E M_{B A}  \tag{5}\\
& t_{A B}=\left[\frac{M_{A B}}{E I} * \frac{L}{2} * \frac{2 L}{3}\right]-\left[\frac{M_{B A}}{E I} * \frac{L}{2} * \frac{L}{3}\right]-F E M_{A B} \tag{6}
\end{align*}
$$

Now substitute (5) and (6) into (4) and (3). We are beginning to find a general equation. After substitution we have...

$$
\begin{align*}
& \theta_{A}-\psi_{A B}=\frac{1}{L}\left[\frac{\left(M_{B A}\right.}{E I} * \frac{L}{2} * \frac{2 L}{3}-\frac{M_{A B}}{E I} * \frac{L}{2} * \frac{L}{3}-F E M_{A B}\right]  \tag{7}\\
& \theta_{B}-\psi_{A B}=\frac{1}{L}\left[\frac{\left(M_{A B}\right.}{E I} * \frac{L}{2} * \frac{2 L}{3}-\frac{M_{B A}}{E I} * \frac{L}{2} * \frac{L}{3}-F E M_{B A}\right] \tag{8}
\end{align*}
$$

Now solve both equations simultaneously for $M_{A B}$ and $M_{B A}$, and simplify the equation.

$$
\begin{align*}
& M_{A B}=\frac{2 E I}{L} *\left[2 \theta_{A}+\theta_{B}-3 \psi_{A B}\right]+F E M_{A B}  \tag{9}\\
& M_{B A}=\frac{2 E I}{L} *\left[2 \theta_{B}+\theta_{A}-3 \psi_{A B}\right]+F E M_{B A} \tag{10}
\end{align*}
$$

We have now finished the derivation of the Slope-Deflection Method for the example given. The general equation follows. This makes sense. We know from moment
distribution that the angle of rotation would yield double the moment at the end it corresponded with. Also you can see that the deflection (sway) of the member has the same amount of influence on the members. Each individual member will receive a general equation. Then we will use a system of linear equations to solve for the unknown rotations an fixed end moments.

$$
\begin{equation*}
M_{x y}=\frac{2 E I}{L} *\left[2 \theta_{x}+\theta_{y}-3 \psi_{x y}\right]+F E M_{x y} \tag{11}
\end{equation*}
$$

If you would like to further simplify this, you can replace the $I / L$ term to a relative flexural stiffness K term. The equation then simplifies to equation 12 below. Although not necessary, you may see it written this way.

$$
\begin{equation*}
M_{x y}=(2 E K) *\left[2 \theta_{x}+\theta_{y}-3 \psi_{x y}\right]+F E M_{x y} \tag{12}
\end{equation*}
$$

Just a side note, fixed connections do not have a assigned $\theta$ value. However, where two members connect will have a angle due to rotation. So, they will receive a $\theta$ value.

## 2 Example

The the example will show how to use the Slope-Deflection Method to solve for internal moments in indeterminate structures. Below is a simple, yet indeterminate frame. Notice that C is not a fixed section, but it is a pinned connection.


Fig. 3 Example Problem Statement.

Step one is to draw the resulting free body diagram with loads.


Fig. 4 Free Body Diagram. Notice that C is a Pin.

Step two is to find all of the fixed end moments in both of our members independently. For Member $A B$ both $M_{A B}$ and $M_{B A}$ are the same fixed end moment. The equation to find these moments can be found in most textbooks or in the AISC Steel Construction Manual. For Member $_{A B}$ the equation is $F E M_{A B}=P * L / 8$. For Member $_{B C}$ the equation to use is $F E M_{B C}=W * L / 24$. The values are 15 Kip ft and $37.5 \mathrm{Kip} * \mathrm{ft}$ respectively. Regardless of the fixity at the ends, the moment always applies. Therefore, the pin at C still receives the allotted moment. Remember to check the signs of the moments. They matter. For all SlopeDeflection method calculations I assume that counterclockwise is positive moment. Most likely, the values of your fixed end moments will alternate sign based off the loading scenario. This is true in our example. Step three is to write the general equations. If there is no rotation, then the equation gets no $\theta$. Likewise if the structure does not sway, (like our example) you will not see a $\psi$ term. So for our example we have four equations. Notice again that equation 13 has no rotational value for the near side because it is a fixed moment. Likewise, 14 does not have a far side rotation, for the same reasoning. Our pin blocks all chance of sway so we have no $\psi$ values.

$$
\begin{align*}
& M_{A B}=\frac{2 E I}{12} *\left[\theta_{B}\right]+15  \tag{13}\\
& M_{B A}=\frac{2 E I}{12} *\left[2 \theta_{B}\right]-15 \tag{14}
\end{align*}
$$

Notice the change of sign between the fixed end moments in equation 11 and 12. Units are in Kip-ft

$$
\begin{align*}
& M_{B C}=\frac{2 E I}{15} *\left[2 \theta_{B}+\theta_{C}\right]-37.5  \tag{15}\\
& M_{C B}=\frac{2 E I}{15} *\left[2 \theta_{C}+\theta_{B}\right]+37.5 \tag{16}
\end{align*}
$$

Step four is to count the number of unknowns vs. the number of equations... You should have counted 4 equations but 6 unknowns... Therefore, we now have to use our heads and look at the structure to find compatibility equations. There are three ways to do this. Sometimes you may be able to reason that two $\theta$ values may be the same, but you must be very cautious taking this approach. An easier approach is to look at the joints, since the moments should just be the opposite
sign. For instance, in our example $M_{B C}$ is the negative moment of $M_{B A}$. Most joints can give you another equation. The last method is to look at the support conditions. Support C is a pin, so it cannot hold a moment. Pins will always give you an equation. For us $M_{C B}=0$. This gives us enough equations to solve. The other two equations we need follow.

$$
\begin{gather*}
-M_{B C}=M_{B A}  \tag{17}\\
M_{C B}=0 \tag{18}
\end{gather*}
$$

Step five is to solve the system of linear equations for the internal moments. I elected to use matrices because I like them for their simplicity and quickness. First, the equations must be rearranged.

$$
\begin{gather*}
-15=\frac{2 E I}{12} *\left[\theta_{B}\right]-M_{A B}  \tag{19}\\
15=\frac{2 E I}{12} *\left[2 \theta_{B}\right]-M_{B A}  \tag{20}\\
37.5=\frac{2 E I}{15} *\left[2 \theta_{B}+\theta_{C}\right]-M_{B C}  \tag{21}\\
-37.5=\frac{2 E I}{15} *\left[2 \theta_{C}+\theta_{B}\right]+M_{C B}  \tag{22}\\
0=M_{B A}+M_{B C}  \tag{23}\\
M_{C B}=0 \tag{24}
\end{gather*}
$$

Now enter these equations into a matrix and solve.

$$
\left|\begin{array}{cccccc}
M_{A B} & M_{B A} & M_{B C} & M_{C B} & \theta_{B} & \theta_{C}  \tag{25}\\
-1 & 0 & 0 & 0 & \frac{1}{6} & 0 \\
0 & -1 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & -1 & 0 & \frac{4}{15} & \frac{2}{15} \\
0 & 0 & 0 & -1 & \frac{2}{15} & \frac{4}{15} \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right| *\left|\begin{array}{c}
15 \\
-15 \\
37.5 \\
-37.5 \\
0 \\
0
\end{array}\right|=\left|\begin{array}{c}
M_{A B} \\
M_{B A} \\
M_{B C} \\
M_{C B} \\
\theta_{B} \\
\theta_{C}
\end{array}\right|
$$

The answers (for the high achievers) are: $M_{A B}=-2.109 \mathrm{kip}-\mathrm{ft}, M_{B A}=40.781$ kip-ft, $M_{B C}=-40.781 \mathrm{kip}-\mathrm{ft}, M_{C B}$ kip-ft, $\theta_{B}=77.334, \theta_{C}=-179.297$. Remember that counterclockwise is a positive moment. We expected that $M_{B C}$ and $M_{B A}$ would be equal and opposite and that $M_{C B}$ would be 0 . It is good that these checks were validated.ANS.

## 3 Common Pitfalls

There are two ways to easily mess up the Slope-Deflection Method. First, make sure your compatibility equations actually work. Sometimes, in a rush, we can write compatibility equations that are not true. You must be sure the the $\theta$ values actually do match or that moments are opposite at a joint. If not, then you cannot assume what we did in the example problems, and you must either find more equations or use a different method for solving indeterminate structures. The second, regards the supports. Remember that fixed connections have no $\theta$ value attached to them. If you add these false $\theta$ values into your calculation, you will result in an incorrect answer. Lastly, fixed end moments only apply to the member that the load is acting on; not adjacent members.

## 4 Conclusion

Hopefully, this helped in understanding the Slope-Deflection Method for finding internal moments in indeterminate structures. With this tool, it becomes easy to calculate moments without calculus or repetitive iterative calculations. This method is effective and highly underrated.

## 5 About the Author

Calvin Kiesewetter is a student at the United Stated Military Academy at West Point. He Studies Civil Engineering. In his free time he enjoys working on problem sets and making jokes about Tom Grady. He is also the only person to be submitted by Dr. Freidenberg more than 20 times, and never once earning a single bonus point. Fortunately for Calvin, he has never lost a pie eating competition (been in 6 ), and holds the record for the most time spent asleep in CME courses. Calvin is moving to Missouri where he will join the Corps of Engineers. Essayons!


Fig. 5 Calvin Kiesewetter.

## 6 Citations

Many of the equations during the derivation of the Slope-Deflection Method were taken from Fundamentals of Structural Analysis $4^{\text {th }}$ Edition by Kenneth M. Leet, Chia-Ming Uang, and Anne M. Gilbert. PGs 466-473.


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