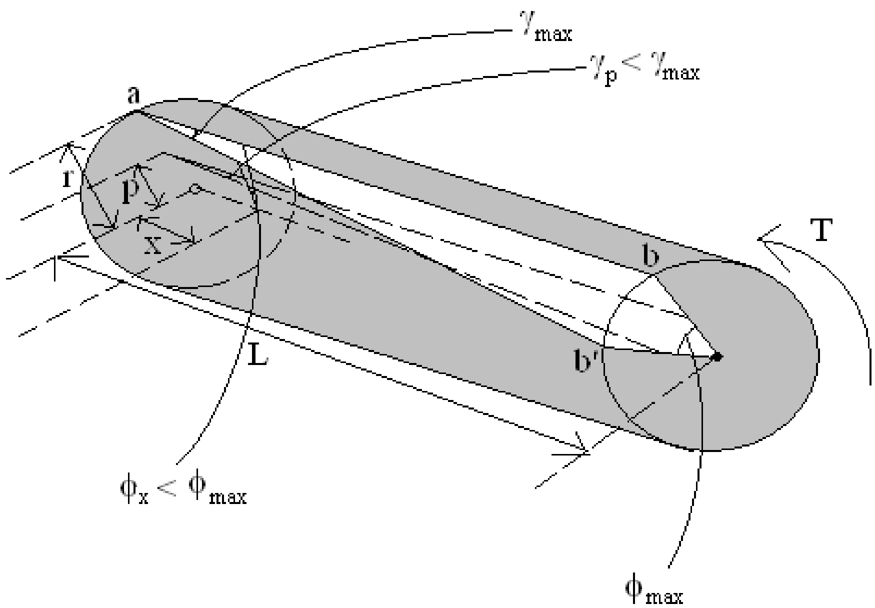
Torsion



Above is a fixed, prismatic beam subjected to a torque at the right end.

ϕ_{max} = angle of twist

ϕ_x depends on distance x from the wall

γ_p depends on distance p from the center

Assume distance bb' is very small and so the arc length bb' is approximately equal to a straight line bb'.

$$\phi_{max} = [\text{fraction of arc length change}](2\pi \text{ radians}) = [\frac{bb'}{2\pi\, r}](2\pi) = \frac{bb'}{r}$$

$$\gamma_{max} = [\frac{bb'}{2\pi(ab)}](2\pi) = \frac{bb'}{ab}$$

We can see that $\gamma_{max} = \frac{r\phi_{max}}{L}$ is really the same expression as above.

Also, $\gamma_p = \frac{p}{r}\gamma_{max} = p\frac{\phi_{max}}{L}$

$\tau = G\gamma$

$$\tau_{max} = G\gamma_{max} = Gr\frac{\phi_{max}}{L} \qquad \tau_p = \frac{p}{r}\tau_{max} = Gp\frac{\phi_{max}}{L}$$

We need to find a relationship between τ and T:

$$T = \int_A \tau_p p\, dA = \int_A (\frac{\tau_{max}}{r} p)p\, dA$$

If polar moment of inertia = $I_p = \int_A p^2 dA$, then:

$T = \frac{\tau_{max}}{r} I_p \Rightarrow \tau_{max} = \frac{Tr}{I_p}$ → general formula for a circular shaft subjected to torsion

$$\boldsymbol{\phi_{max} = \frac{\tau_{max} L}{Gr} = \frac{TL}{GI_p} \qquad \gamma_{max} = \frac{r\phi_{max}}{L} = \frac{\tau_{max}}{G} = \frac{Tr}{GI_p}}$$

ϕ_{max} is often just written ϕ

note: $\phi_x = \frac{x}{L}\phi_{max} = \frac{\tau_{max} x}{Gr}$ (also note similarity of ϕ_{max} above to $\delta = \frac{PL}{EA}$)

Solid Bar:

$$T = \frac{\tau_{max}}{r} \int_{\phi=0}^{2\pi} \int_{p=0}^{r} p^2 p \, dp \, d\theta = \left(\frac{\tau_{max}}{r}\right)\frac{\pi r^4}{2} = \frac{\tau_{max} \pi d^3}{16} \Rightarrow$$

$$\boldsymbol{\tau_{max} = \frac{16T}{\pi d^3}} \textbf{ (solid shaft)}$$

note: recall from calculus that the extra p in the integrand is just an extra polar integration factor

Hollow Tube:

$$T = \frac{\tau_{max}}{r} \int_{\theta=0}^{2\pi} \int_{p=r1}^{r2} p^2 p \, dp \, d\theta = \left(\frac{\tau_{max}}{r}\right)\frac{\pi}{2}(r_2^4 - r_1^4) = \left(\frac{\tau_{max}}{2}\right)\frac{\pi}{32}(d_2^4 - d_1^4) \Rightarrow$$

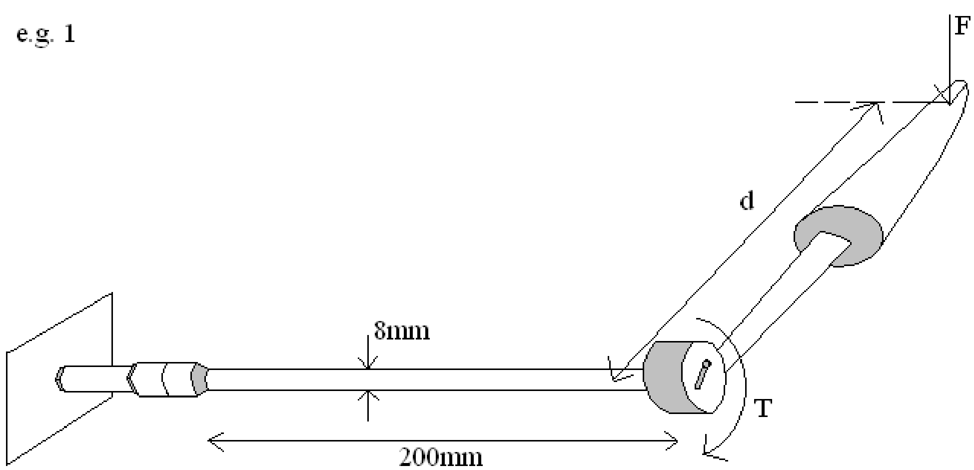
$$\boldsymbol{\tau_{max} = \frac{16Td_2}{\pi(d_2^4 - d_1^4)}} \quad \textbf{(tube)}$$

e.g. 1

Given: Socket wrench transmits torque to a stuck bolt.

$\tau_{allowable} = 460MPa \quad G = 78GPa$ *for the 8mm diameter, solid shaft shown*

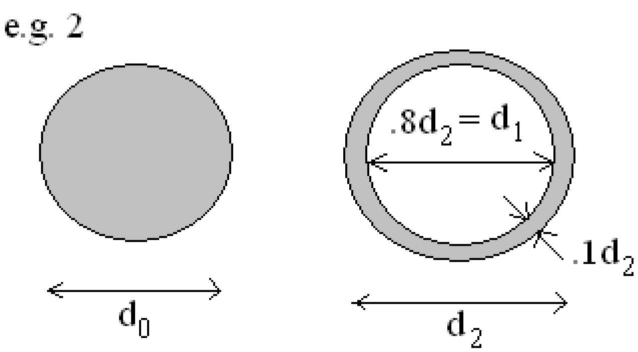
Find: T_{max} *and* ϕ_{max} *for this allowable torque value*



$$\tau_{max} = \frac{16T}{\pi d^3} \quad T_{max} = \frac{(\tau_{allowable})\pi(8x10^{-3})^3}{16} = \mathbf{46.25N*m} \qquad (F_{max} = \frac{T_{max}}{d})$$

$$\phi = \frac{\tau_{max} L}{Gr} = \frac{(\tau_{allowable})(200x10^{-3})}{(78x10^9)(\frac{8}{2}x10^{-3})} = .29\,Rad \quad or\ (.29\,Rad)(\frac{180°}{\pi\,Rad}) = \mathbf{16.6°}$$

e.g. 2
Given: Either a solid or a hollow steel shaft is to be manufactured,
$T_{max} = 1200N*m \quad \tau_{allowable} = 40MPa \quad$ *Thickness of hollow shaft* $= .1\,d_2$
Find: $(d_0)_{min}, (d_2)_{min}$, *and the ratio of material usage for the hollow shaft versus the solid shaft.*



Solid:

$$\tau_{max} = \frac{16T}{\pi d^3} \Rightarrow (d_0)_{min} = \sqrt[3]{\frac{16T_{max}}{\pi\,\tau_{allowable}}} = \mathbf{53.5mm}$$

Tube:

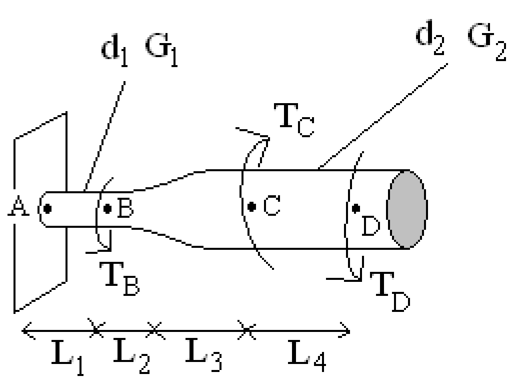
$$\tau_{max} = \frac{16Td_2}{\pi[d_2^{\,4} - (.8d_2)^4]} \Rightarrow$$

$$(d_2)_{min} = \sqrt[3]{\frac{16T_{max}}{(\tau_{allowable})\pi(1-.8^4)}} = \mathbf{63.7mm}$$

Since both shafts are the same density and length, the ratio of weights = the ratio of volumes = the ratio of areas:

$$\frac{A_{hollow}}{A_{solid}} = \frac{\pi/4\,(d_2^{\,2} - d_1^{\,2})}{\pi/4\,d_0^{\,2}} = \mathbf{.47}$$

The hollow shaft has a larger diameter, but only uses 47% as much material as the solid shaft. Hollow shafts are more efficient.



$$\phi_1 = \frac{T_1 L_1}{G_1 I_{p1}} \quad \phi_2 = \frac{T_2 L_2}{G_1 I_{p1}} \quad \phi_3 = \frac{T_3 L_3}{G_2 I_{p2}} \quad \phi_4 = \frac{T_4 L_4}{G_2 I_{p2}}$$

$$\phi_{total} = \sum \phi$$

T_1, T_2, T_3, T_4 *are the internal torques within sections* L_1, L_2, L_3, L_4, *respectively, which can be found from drawing free body diagrams as was done for the axial case in the previous section on Hooke's Law.*

Deformation of tapered bars in torsion

Continuously varying torques/dimensions;

$$d\phi = \frac{T(x)\,d(x)}{GI_p(x)} \qquad \phi = \int_0^L \frac{T(x)}{GI_p(x)}dx$$

note: satisfactory as long as angle of taper is less than 10°

$I_p(x)$ determined from d(x), where d is the diameter

note: A shaft in torsion has a normal stress. If $\sigma_{allowable}$ for a material is equal to, or less than $\tau_{allowable}$, then the design for the shaft in torsion is controlled by σ. And, it will fail along a 45° axis. (proof – section on Mohr's Circle later in this chapter) (e.g. chalk)

note: Similarly, a member under axial load has a shear stress. If $\tau_{allowable}$ for a material is equal to, or less than $\frac{1}{2}(\sigma_{allowable})$, then the design for the axially loaded member is controlled by τ. And, it will fail along a 45° axis. (proof- section on Mohr's Circle) (e.g. concrete)

Gere, James M. Mechanics of Materials: Sixth Edition. Brooks/Cole. Belmont, CA 2004.

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