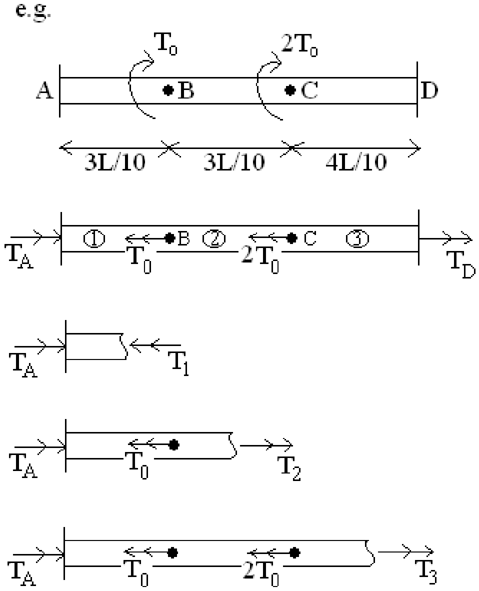
Simple statically indeterminate system (torsion)

e.g.
Given: Circular bar with fixed (rigid) ends shown.
Find: Support reactions and ϕ_{max}.

e.g.



$$\overset{\rightarrow\rightarrow}{+}\sum T_A + T_D - T_0 - 2T_0 = 0 \ \textit{(eq of equilibrium)}$$

From the three FBD's to the left, we can see that:

$T_1 = T_A \quad T_2 = T_0 - T_A \quad T_3 = 3T_0 - T_A$

So,

$$\phi_1 = \frac{(T_A)^{3L}/_{10}}{GI_p} \quad \phi_2 = \frac{(T_0 - T_A)^{3L}/_{10}}{GI_p}$$

$$\phi_3 = \frac{(3T_0 - T_A)^{4L}/_{10}}{GI_p}$$

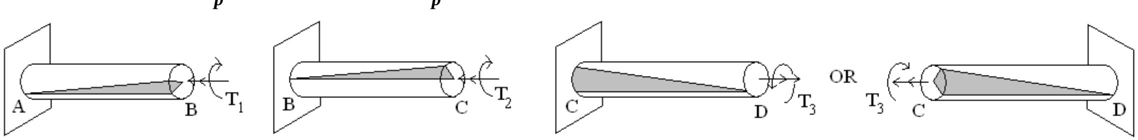
$$\overset{\rightarrow\rightarrow}{+}\sum \frac{-(T_A)^{3L}/_{10}}{GI_p} + \frac{(T_0 - T_A)^{3L}/_{10}}{GI_p} + \frac{(3T_0 - T_A)^{4L}/_{10}}{GI_p} = 0$$

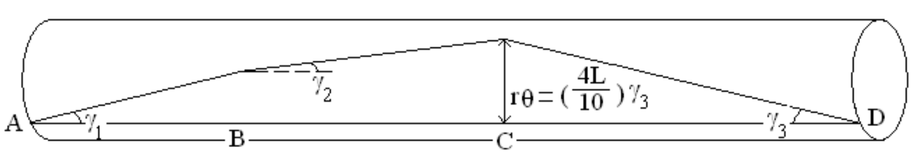
(eq of compatibility)

2 eq, 2 unknowns $\Rightarrow \boldsymbol{T_A = {}^{3T_0}/_2} \quad \boldsymbol{T_D = 3T_0 - \frac{3T_0}{2} = {}^{3T_0}/_2}$

$$\phi_1 = \frac{({}^{3T_0}/_2)^{3L}/_{10}}{GI_p} = \frac{9T_0L}{20GI_p} \quad \phi_2 = \frac{(T_0 - {}^{3T_0}/_2)^{3L}/_{10}}{GI_p} = \frac{-3T_0L}{20GI_p} \Rightarrow \phi_2 = \frac{3T_0L}{20GI_p} \longleftarrow\!\!\longleftarrow$$

$$\boldsymbol{\phi_3 = \frac{(3T_0 - {}^{3T_0}/_2)^{4L}/_{10}}{GI_p} = \frac{3T_0L}{5GI_p} = \phi_{max}}$$





Quick summation of arc-lengths $r\gamma$ *prove that*

$({}^{3L}/_{10})\gamma_1 + ({}^{3L}/_{10})\gamma_2$

$= (4L/10)\gamma_3$

note: Direction of ϕ for each segment should be consistent with the direction of torques on free-body diagrams for each segment. Incorrect guess will simply result in negative values for ϕ.

note: As always with design, the allowable stress (τ_{allow}) and the force (T) are known, and we want to minimize the area A. Axial, bearing, and direct shear stresses are related to A, so we can easily minimize A. Shear stress for a solid shaft in torsion is not related to A, but it is related to d, so we can easily minimize A. The stress for a hollow tube in torsion, however, is not related to A and it depends on more than one dimension (d_1 and d_2). There are thus three unknowns (d_1, d_2, and A) and two equations ($A = \frac{\pi}{4}(d_2^{\,2} - d_1^{\,2})$, $\tau_{allow} = \frac{16Td_2}{\pi(d_2^{\,4} - d_1^{\,4})}$). One might be tempted to use $\frac{dA}{dx} = 0$ for a third equation, but there is no local minimum. It turns out, not surprisingly, that $A \to 0$ as $d_2 \to \infty$ and $d_1 \to d_2$. The best method, for this particular case, would be to use a table of common tube sizes and pick the tube with the smallest area in which $\tau \le \tau_{allow}$. In engineering practice, methods that utilize tables are often used, particularly for the selection of timber and steel section sizes for flexure, which we will learn about next.

Gere, James M. Mechanics of Materials: Sixth Edition. Brooks/Cole. Belmont, CA 2004.

Lee, Vincent. Lecturer. University of Southern California. CE225. Spring 2005.