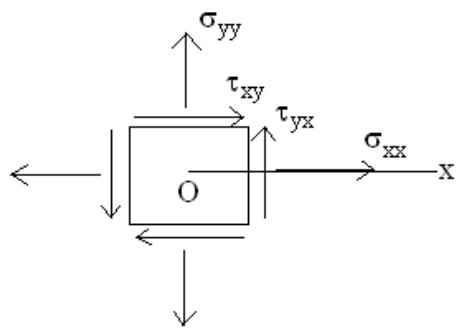
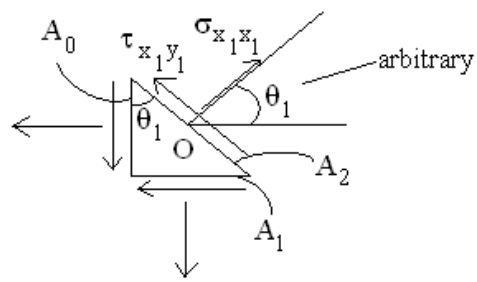
Principal stresses



Stresses are positive if <u>positive</u> face – <u>positive</u> direction or <u>negative</u> face – <u>negative</u> direction.
(All stresses shown are positive with respect to these axes)

$$\cos\theta_1 = \frac{A_0}{A_2} \Rightarrow A_2 = A_0 \sec\theta_1$$

$$\tan\theta_1 = \frac{A_1}{A_0} \Rightarrow A_1 = A_0 \tan\theta_1$$



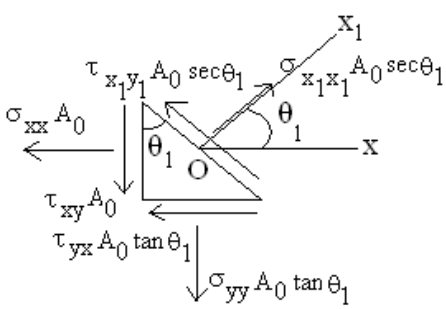
stresses

From $\sum F_x = 0$ and $\sum F_y = 0$,

$$\boldsymbol{\sigma_{x1x1} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta_1 + \tau_{xy}\sin 2\theta_1}$$

$$\boldsymbol{\tau_{x1y1} = \frac{-(\sigma_{xx} - \sigma_{yy})}{2}\sin 2\theta_1 + \tau_{xy}\cos 2\theta_1}$$

$\theta_1 \; \varepsilon \; [0°, 180°]$

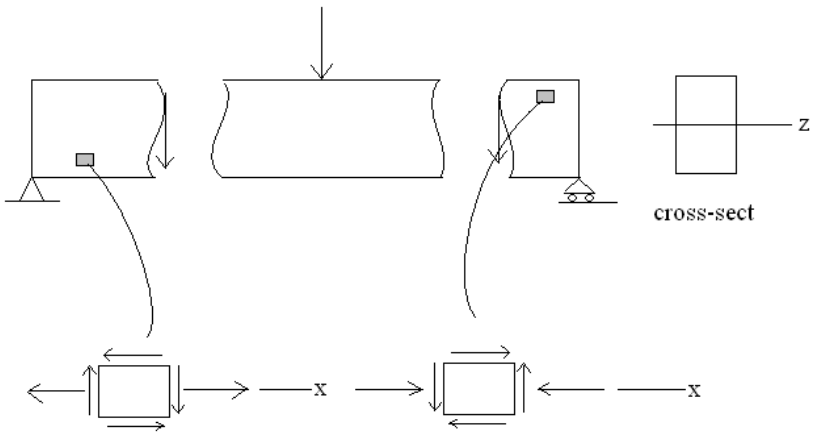


forces

These are the "transformation equations" for plane stress. σ_{max} and τ_{max} from these equations are the <u>true</u> maximum stresses in a beam (except for the special case where they occur out-of-plane). σ_{max} and τ_{max} may occur at a location of $(\sigma_{xx})_{max}, (\sigma_{yy})_{max}, (\tau_{xy})_{max}$, or may occur at a location where none of the above are maximized.

note: σ_{yy} for a given region in a beam is the distributed load q divided by the cross-sect thickness t at that location.

note: σ_{yy} usually compressive (negative in the above equations) since our distributed loads act downward. σ_{xx} direction determined from bending stress and external axial load ($\frac{P}{A} + \frac{My}{I}$). τ direction determined from inspection of the internal vertical equilibrium (NOT SFD) (see below).



τ reversed because location is to the <u>right</u> of the point load.

σ reversed because section chosen is in <u>upper</u> portion of beam.

note: The beams of chapter five usually contain all three forces. Since σ_{yy} depends on x (distance along beam) and y (in relation to neutral axis), σ_{xx} depends on x and y, τ_{xy} depends on x and y, and θ_1 also varies between 0 and 180°, finding the exact location and angle of σ_{max} and τ_{max} can usually only be approached through trial and error using the transformation equations. For design, transformation equations are accounted for in the safety factor, but can be checked as follows.

<u>Principal Angles</u>

The following is useful assuming that a location O within a beam has been chosen and $\tau_{xy}, \sigma_{xx}, \sigma_{yy}$ are known.

From $\dfrac{d\sigma_{x1x1}}{d\theta_1} = 0$, $\mathbf{\tan 2(\theta_p) = \dfrac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}}$ (critical angles for normal – "principal stress")

Two solutions: $\theta_p \in [0, 90°]$ and $\theta_p \in [90, 180°]$ which correspond to θ_{p1} and θ_{p2} though not necessarily in that order. θ_{p1} and θ_{p2} differ by 90°.

$$\boldsymbol{\sigma_{xp1xp1} = (\sigma_{x1x1})_{max} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\ 2}}} \quad \text{(skipped work)}$$

$$\boldsymbol{\sigma_{xp2xp2} = (\sigma_{x1x1})_{min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\ 2}}}$$

(could be greater <u>magnitude</u> than $(\sigma_{x1x1})_{"max"}$)

note: $\tau_{xp1xp1} = \tau_{xp2xp2} = 0$ (proof Mohr's Circle – see next section)

note: The true min and max normal stress <u>could</u> be located "out-of-plane" (not calculated)

From $\dfrac{d\tau_{x1y1}}{d\theta_1} = 0$, $\mathbf{\tan 2(\theta_s) = \dfrac{-(\sigma_{xx} - \sigma_{yy})}{2\tau_{xy}}}$ (critical angles for shear stress)

Two solutions: $\theta_s \in [0, 90°]$ and $\theta_s \in [90, 180°]$ which correspond to θ_{s1} and θ_{s2} though not necessarily in that order. θ_{s1} and θ_{s2} differ by 90°.

$$\boldsymbol{\tau_{xs1ys1} = (\tau_{x1y1})_{max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\ 2}}} \quad \textbf{OR} \quad \boldsymbol{\frac{\sigma_{xp1xp1} - \sigma_{xp2xp2}}{2}}$$

$$\tau_{xs2ys2} = (\tau_{x1y1})_{min} = -(\tau_{x1y1})_{max}$$

note: $\sigma_{ave} = \dfrac{\sigma_{xx} + \sigma_{yy}}{2} = \sigma_{xs1xs1} = \sigma_{xs2xs2}$ (Proof Mohr's Circle – see next section)

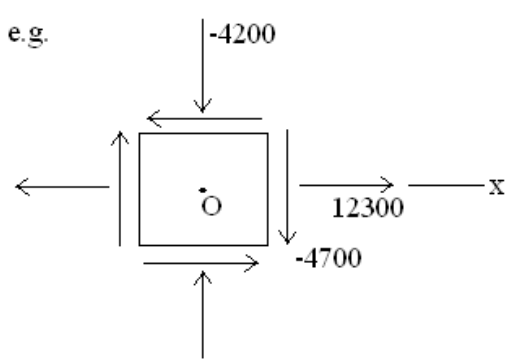
note: $\theta_{s1} = \theta_{p1} - 45°$ (Proof Mohr's Circle)

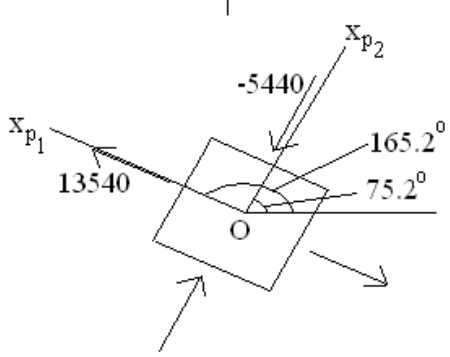
note: The true min and max shear stress is located out-of-plane if σ_{xp1xp1} and σ_{xp2xp2} have the same sign: [$(\tau_{max/min})_{about\ xp1} = \pm\dfrac{\sigma_{xp2xp2}}{2}$ and $(\tau_{max/min})_{about\ xp2} = \pm\dfrac{\sigma_{xp1xp1}}{2}$].

e.g.

Given: $\sigma_{xx} = 12300\,psi \quad \sigma_{yy} = -4200\,psi \quad \tau_{xy} = -4700\,psi$

Find: $\sigma_{xp1xp1}, \sigma_{xp2xp2}, \tau_{xs1ys1}, \tau_{xs2ys2}$





$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \Rightarrow 2\theta_p = -29.67° = 330.33°$$

$$\theta_p = 165.2° \text{ and } 75.2°$$

$$\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2(165.2) + \tau_{xy}\sin 2(165.2)$$

$= \mathbf{13540\ psi} = \sigma_{xp1xp1}$

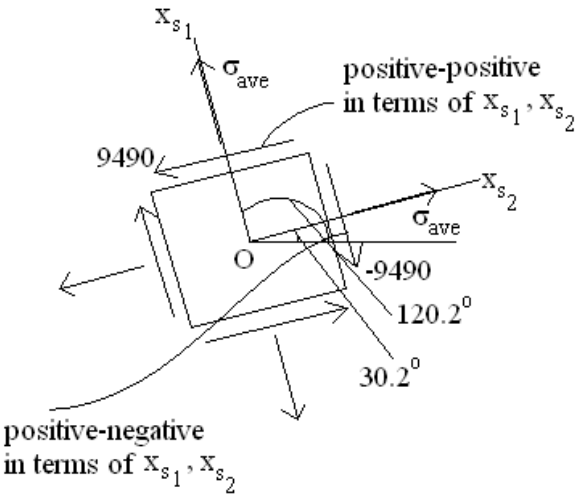
$$\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2(75.2) + \tau_{xy}\sin 2(75.2)$$

$= \mathbf{-5440\ psi} = \sigma_{xp2xp2}$

check:

$$\frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

= -5440 and 13540, as expected



$$\tan 2\theta_s = \frac{-(\sigma_{xx} - \sigma_{yy})}{2\tau_{xy}} \Rightarrow 2\theta_s = 60.4°$$

$$\theta_s = 30.2° \text{ and } 120.2°$$

$$\frac{-(\sigma_{xx} - \sigma_{yy})}{2} \sin 2(30.2) + \tau_{xy} \cos 2(30.2)$$

$$= \textbf{-9490 psi} = \tau_{xs2ys2}$$

$$\frac{-(\sigma_{xx} - \sigma_{yy})}{2} \sin 2(120.2) + \tau_{xy} \cos 2(120.2)$$

$$= \textbf{9490 psi} = \tau_{xs1ys1}$$

$$\text{check: } \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\ 2}} = \pm 9490$$

note: $\theta_{s1} = \theta_{p1} - 45° = 165.2° - 45° = 120.2°$

note: $\tau_{xs1ys1} = -\tau_{xs2ys2} = \dfrac{\sigma_{xp1xp1} - \sigma_{xp2xp2}}{2} = \dfrac{13540 - (-5440)}{2} = 9490$

note: $\tau_{xp1yp1} = \tau_{xp2yp2} = 0 \text{ and } \sigma_{xs1xs1} = \sigma_{xs2xs2} = \sigma_{ave}$ could also be shown easily