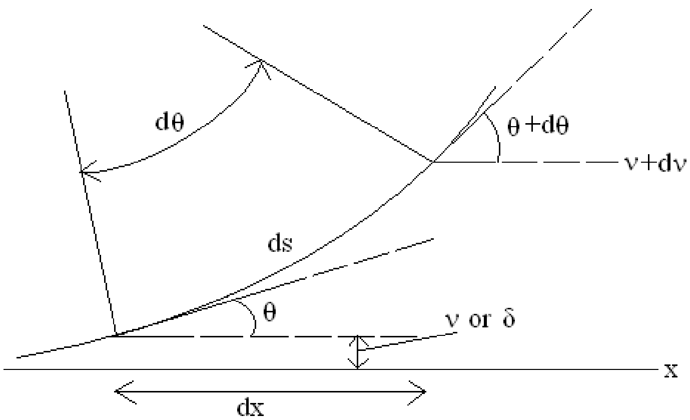
Beam deflections and rotations



$$\kappa = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \text{ and } \frac{d\upsilon}{dx} = \tan\theta \approx \theta$$

So, $$\frac{d\theta}{dx} = \frac{d^2\upsilon}{dx^2} = \kappa = \frac{M}{EI}$$

$$\Rightarrow EI\frac{d^2\upsilon}{dx^2} = M$$

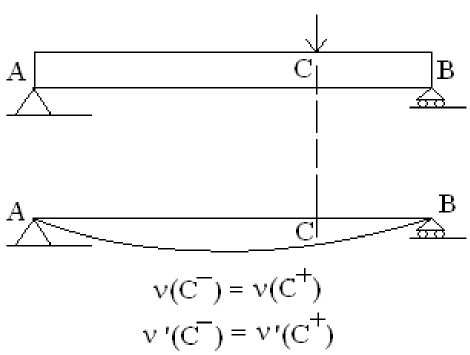
Substitute BMD function(s) for M, and integrate.

$$\mathbf{EI\upsilon'''' = -q \qquad EI\upsilon''' = V \qquad EI\upsilon'' = M \qquad EI\upsilon' = EI\theta \qquad EI\upsilon = EI\delta}$$

q, V, M should be non-zero constant, or functions of x
(If non-prismatic beam, then I also depends on x)

note: small deflections only

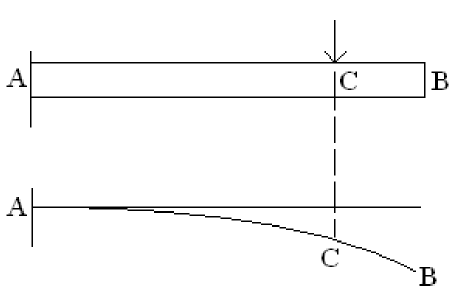
Solving the second-order bending moment equation $EI\upsilon'' = M$, yields two constants of integration (for each segment of a beam). We need two sets of initial conditions (for each segment). There are always enough to choose from, if the system is statically determinate:



Simply-supported beam shown:

- boundary conditions:
  - $\upsilon(A) = 0$
  - $\upsilon(B) = 0$
- continuity conditions:
  - $\upsilon(C^-) = \upsilon(C^+)$
  - $\upsilon'(C^-) = \upsilon'(C^+)$
- symmetry conditions:
  none

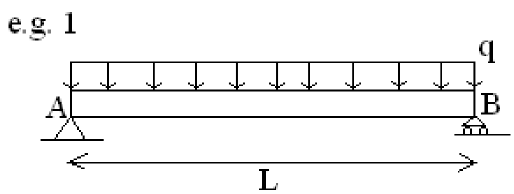
Cantilevered beam shown:



- boundary conditions:
  - $\upsilon(A) = 0$
  - $\upsilon'(A) = 0$

- continuity conditions:
    - $\upsilon(C^-) = \upsilon(C^+)$
    - $\upsilon'(C^-) = \upsilon'(C^+)$
- symmetry conditions:
  none

e.g. 1
Find: Deflection curve $\upsilon, \delta_{max}, \theta_{max}$.



$$M = \frac{qL}{2}(x) - qx(\frac{x}{2}) = \frac{qLx}{2} - \frac{qx^2}{2} \text{ (skipped work)}$$

$$EI\upsilon'' = \frac{qLx}{2} - \frac{qx^2}{2}$$

$$EI\int \upsilon''\,dx = EI\upsilon' = \int \frac{qLx}{2}dx - \int \frac{qx^2}{2}dx$$

$$= \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1$$

$$EI\int \upsilon'\,dx = EI\upsilon = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1 x + C_2$$

symmetry condition: $\upsilon'(\frac{L}{2}) = 0$

$$0 = \frac{qL}{4}(\frac{L}{2})^2 - \frac{q}{6}(\frac{L}{2})^3 + C_1 \Rightarrow C_1 = -\frac{qL^3}{24}$$

$$EI\upsilon = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3 x}{24} + C_2$$

boundary condition: $\upsilon(L) = 0 \text{ or } \upsilon(0) = 0$

$$C_2 = 0$$

$$\boldsymbol{\upsilon = -\frac{qx}{24EI}(L^3 - 2Lx^2 + x^3)} \quad \delta_{max} \text{ located at } \upsilon' = 0$$

$$\frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} = 0 \Rightarrow x = \frac{L}{2} \text{ as expected}$$

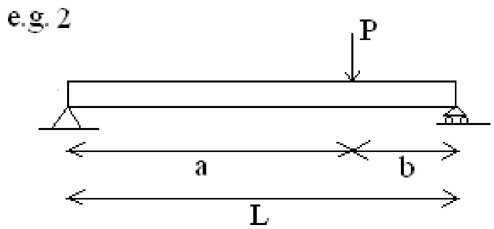
$$\delta_{max} = \upsilon(\frac{L}{2}) = \boldsymbol{\frac{5qL^4}{384EI}} \quad \theta_{max} \text{ located at } \upsilon'' = 0$$

$$\frac{qLx}{2} - \frac{qx^2}{2} = 0 \Rightarrow x = 0 \text{ or } L \text{ as expected}$$

$$\theta_{max} = \upsilon'(L) = |\upsilon'(0)| = \boldsymbol{\frac{qL^3}{24EI}}$$

e.g. 2

Find: $\upsilon_1 \; \varepsilon \; 0 \le x \le a^- \; , \upsilon_2 \; \varepsilon \; a^+ \le x \le L \; , \theta_1 \; , \theta_2 \; , \delta_{max}$



$$0 \le x \le a^- : M = \frac{Pbx}{L} \quad \text{(skipped work)}$$

$$a^+ \le x \le L : M = \frac{Pbx}{L} - P(x-a) \quad \text{(skipped work)}$$

$$EI\upsilon_1'' = \frac{Pbx}{L} \qquad\qquad EI\upsilon_2'' = \frac{Pbx}{L} - P(x-a)$$

$$EI\upsilon_1' = \frac{Pbx^2}{2L} + C_1 \qquad\qquad EI\upsilon_2' = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2$$

$$EI\upsilon_1 = \frac{Pbx^3}{6L} + C_1x + C_3 \qquad\qquad EI\upsilon_2 = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2x + C_4$$

continuity condition: $\upsilon_1'(a^-) = \upsilon_2'(a^+)$

$$\frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} - \frac{P(a-a)^2}{2} + C_2 \quad \text{cancelling terms} \Rightarrow C_1 = C_2$$

continuity condition: $\upsilon_1(a^-) = \upsilon_2(a^+)$

$$\frac{Pba^3}{6L} + C_1a + C_3 = \frac{Pba^3}{6L} - \frac{P(a-a)^3}{6} + C_2a + C_4 \quad \text{cancelling terms} \Rightarrow C_3 = C_4$$

boundary condition: $\upsilon_1(0) = 0$

$$0 = \frac{Pb(0)^3}{6L} + C_1(0) + C_3 \Rightarrow C_3 = 0$$

boundary condition: $\upsilon_2(L) = 0$

$$0 = \frac{Pb(L)^3}{6L} - \frac{P(L-a)^3}{6} + C_2L + C_4 \quad C_4 = 0 \Rightarrow C_2 = \frac{-Pb(L^2-b^2)}{6L}$$

$0 \le x \le a^- :$ $\qquad\qquad$ $a^+ \le x \le L :$

$$\boldsymbol{\upsilon_1 = \frac{-Pbx}{6LEI}(L^2 - b^2 - x^2)} \qquad \boldsymbol{\upsilon_2 = \frac{-Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI}}$$

$$\boldsymbol{\theta_1 = \upsilon_1' = \frac{-Pb}{6LEI}(L^2 - b^2 - 3x^2)} \qquad \boldsymbol{\theta_2 = \frac{-Pb}{6LEI}(L^2 - b^2 - 3x^2) - \frac{P(x-a)^2}{2EI}}$$

For a > b, δ_{max} *obviously* $\varepsilon \; (0, a^-)$.

$$\delta_{max} \text{ at } \upsilon_1' = 0 \Rightarrow x = \sqrt{\frac{L^2 - b^2}{3}} \qquad \delta_{max} = \upsilon_1\left(\sqrt{\frac{L^2-b^2}{3}}\right) = \boldsymbol{\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}\,LEI} \; (a \ge b)}$$

note: The special case method for finding $\delta_{midpoint}$ *in the flexure derivation can still be used.*

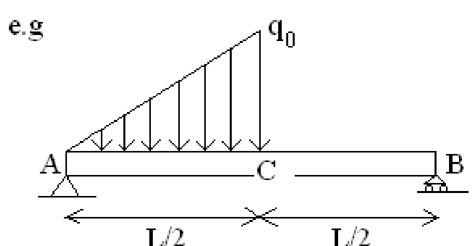
note: Starting with the bending moment equation always works.

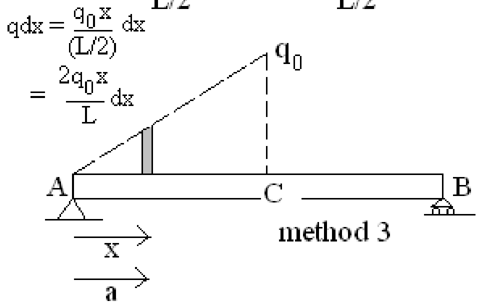
<u>Superposition</u>

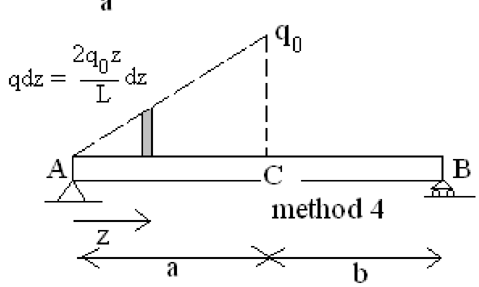
For beams with common uniform loads AND point loads, where $\upsilon(x)$ and $\theta(x)$ can be looked up in a table for the cases where each type of loading is acting alone, $\upsilon_{total} = \sum \upsilon$ and $\theta_{total} = \sum \theta$. Values can be found at specific points, or general (in terms of x) formulas can be found.
Superposition can provide a useful shortcut for unusual loads too. But for these loads, it is usually NOT possible to obtain a general formula $\upsilon(x)$ and $\theta(x)$ for the whole beam because point load formulas (which are different for the left side of the load versus the right side) must be summed, and the shortcut involves an infinite number of point loads. (see next example).

e.g.
Find: δ_c

Method 1: find M(x) and solve $EI\upsilon''$
Method 2: find $-q(x)$ *and solve* $EI\upsilon'''' \ \varepsilon\ [A, C]$
Method 3: point load midpoint deflection formula (tabulated in the appendix of many







textbooks): $\frac{Pa}{48EI}(3L^2 - 4a^2) \quad b \geq a$ (note: this equation works for all points under the load, i.e. between A and C)
For an arbitrary point under the triangular load, the force P = qdx = $\frac{2q_0 x}{L}dx$ and the distance "a" is "x". The deflection at C is the sum of the deflections caused by each infinitesimal force.

$$\delta_c = \int_0^{L/2} \frac{qx}{48EI}(3L^2 - 4x^2)dx = \boldsymbol{\frac{q_0 L^4}{240EI}}$$

Method 4: point load deflection formula for $a \leq x \leq L$ *from a previous example:*

"a" is "z" and "b" is "L-z"

$$\frac{-Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI}$$

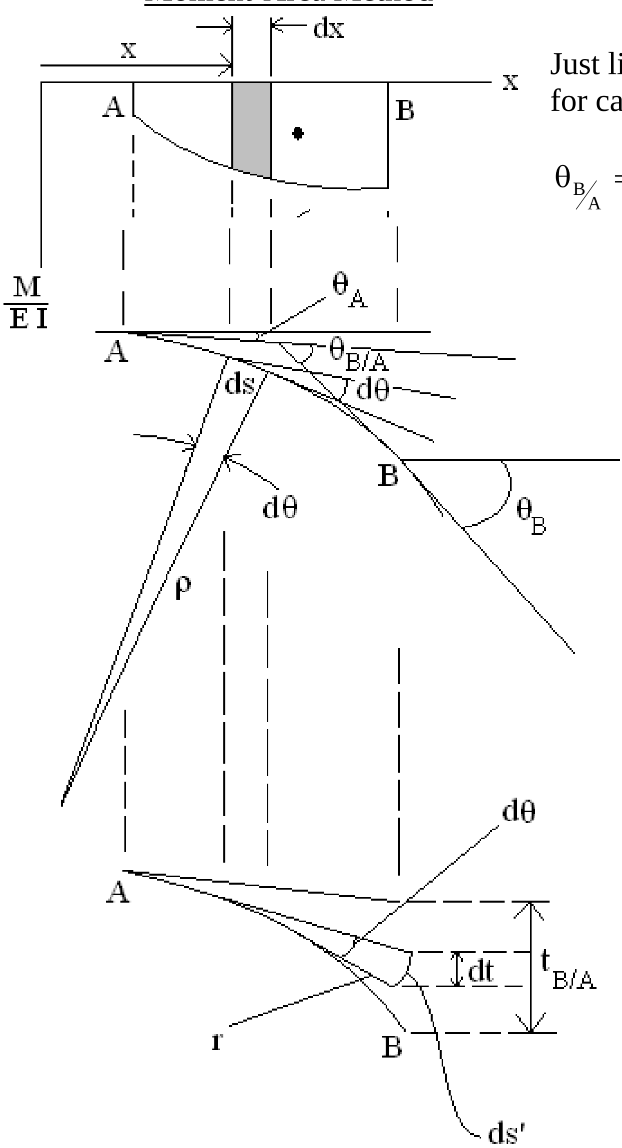
$$P = qdx = \frac{2q_0 z}{L}dz$$

$$\upsilon(x)\ \varepsilon\ [C,B] = \int \frac{-q_0 z(L-z)(x)}{3L^2 EI}(L^2 - (L-z)^2 - x^2)$$
$$-\frac{q_0 z(x-z)^3}{3LEI}dz = \frac{q_0 L(3L^3 - 43L^2 x + 60Lx^2 - 20x^3)}{1440EI}$$
$$\upsilon(L/2) = \frac{\boldsymbol{-q_0 L^4}}{\boldsymbol{240EI}}$$

note: If the triangular load starts at a distance "k" away from A, then the lower limit of integration would be k.

note: There is no <u>easy</u> way to obtain a <u>general</u> formula for the beam, which includes $\upsilon(x)\ \varepsilon\ [A,C]$, because under the triangular load, the location of υ is to the left of some of the "point loads" and to the right of others (two separate formulas).

<u>Moment-Area Method</u>



Just like load equations, this method is particularly useful for cantilevered beams.

$$\theta_{B/A} = \theta_B - \theta_A$$

$$\frac{d\theta}{ds} \approx \frac{d\theta}{dx} = \frac{M}{EI} \Rightarrow d\theta = \frac{Mdx}{EI}$$

$$\theta_{B/A} = \int_A^B d\theta$$

$$\boldsymbol{\theta_B - \theta_A = \int_A^B \frac{Mdx}{EI}} \quad \text{(cantilevered beam)}$$

$$t_{B/A} = t_B - t_A$$

$$\frac{d\theta}{2\pi}(2\pi r) = ds'$$

$ds' = r\,d\theta$

$ds' \approx dt$ and $r \approx x_1$

$$dt = x_1 d\theta = x_1 \frac{Mdx}{EI}$$

(x_1 with respect to B as shown)

note: Although it may not be obvious, the assumptions made here are the same as $ds \approx dx$ and $\tan\theta \approx \theta$.

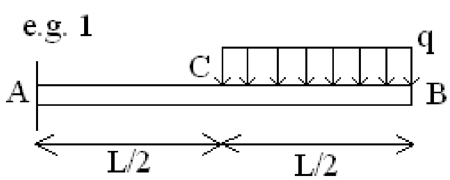
$$t_{B/A} = \int_A^B dt$$

$$\mathbf{t_B - t_A = \int_A^B \frac{x_1 M dx}{EI}} \text{ (cantilevered beam)}$$

If A = fixed end, then $t_B = \delta_B$, $t_A = (x_B - x_A)\theta_A = 0$ (cantilevered)

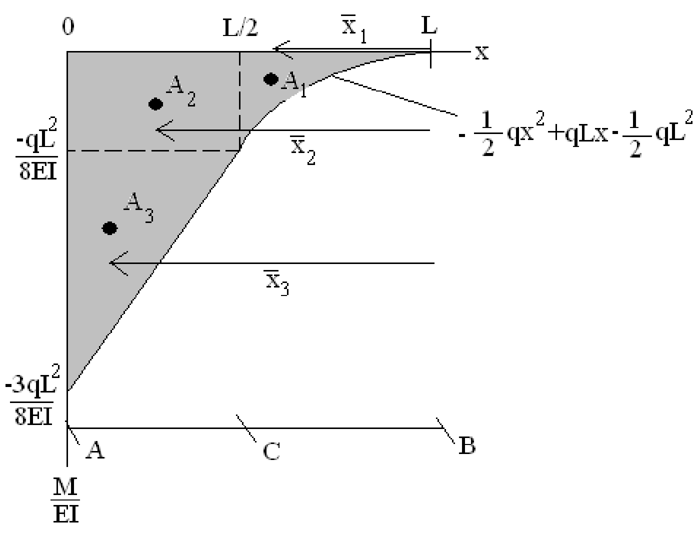
Note: For simply supported beams, since the concavity is reversed compared to cantilevered beams, θ is oriented differently, and t is on the opposite side of the deflection curve from δ. (see second e.x.)

e.g. 1
Find: δ_B



Method 1: find M(x) and solve $EI\upsilon''$
Method 2: find –q(x) and try to solve $EI\upsilon'''' \; \varepsilon \; [C, B]$
<u>Method 3: point load end point deflection formula and superposition</u>
Method 4: point load deflection formula for cantilevered beam for $a \le x \le L$ *and superposition*

Method 5: use area under $\frac{M}{EI}$ *diagram*



$$\overline{x_2}A_2 = \frac{L}{2}\left(\frac{qL^2}{8EI}\right)\left(\frac{L}{2}+\frac{L}{4}\right) = \frac{qL^3}{16EI}\left(\frac{3L}{4}\right)$$

$$\overline{x_3}A_3 = \frac{1}{2}\left(\frac{L}{2}\right)\left(\frac{3qL^2}{8EI} - \frac{qL^2}{8EI}\right)\left[\frac{L}{2} + \frac{2}{3}\left(\frac{L}{2}\right)\right]$$

$$= \frac{qL^3}{16EI}\left(\frac{5L}{6}\right)$$

$$\int_A^C dt = -\left[\frac{qL^3}{16EI}\left(\frac{3L}{4}\right) + \frac{qL^3}{16EI}\left(\frac{5L}{6}\right)\right]$$

$$\int_C^B dt = \int_C^B \frac{x_1 M}{EI} dx \quad \text{where } x_1 = L - x$$

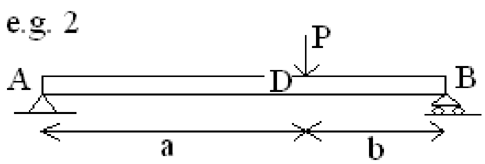
$$\text{and } \; M = -\frac{1}{2}qx^2 + qLx - \frac{1}{2}qL^2$$

$$\Rightarrow \int_C^B \frac{x_1 M}{EI} dx = \int_{L/2}^{L} \frac{(L-x)(-\frac{1}{2}qx^2 + qLx - \frac{1}{2}qL^2)}{EI} dx = \frac{-qL^4}{128EI}$$

$\delta_B - \delta_A = \delta_B$ since $\delta_A = 0$.

$$\Rightarrow \delta_B = \int_A^B dt = \frac{qL^3}{16EI}\left(\frac{3L}{4}\right) + \frac{qL^3}{16EI}\left(\frac{5L}{6}\right) + \frac{qL^4}{128EI} = \boldsymbol{\frac{41qL^4}{384EI}}$$

e.g. 2
Find: δ_D

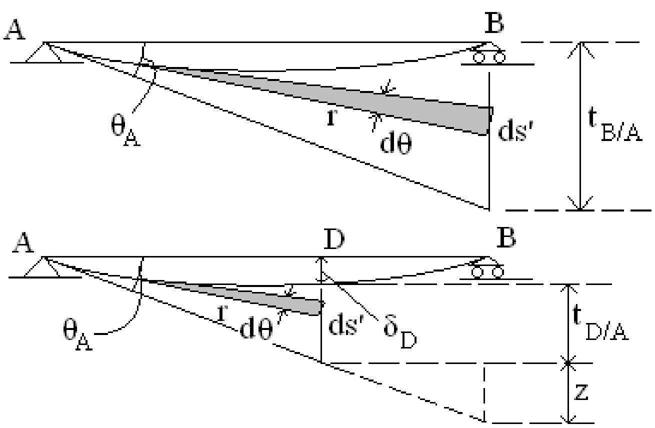


$ds' = r d\theta$

$ds' \approx dt$ and $r \approx x_1$

Just as in the derivation for the cantilevered beam.

$$t_{B/A} = A_1 \bar{x}_1 = \frac{Pab}{2EI}\left(\frac{L+b}{3}\right) = \frac{Pab}{6EI}(L+b)$$



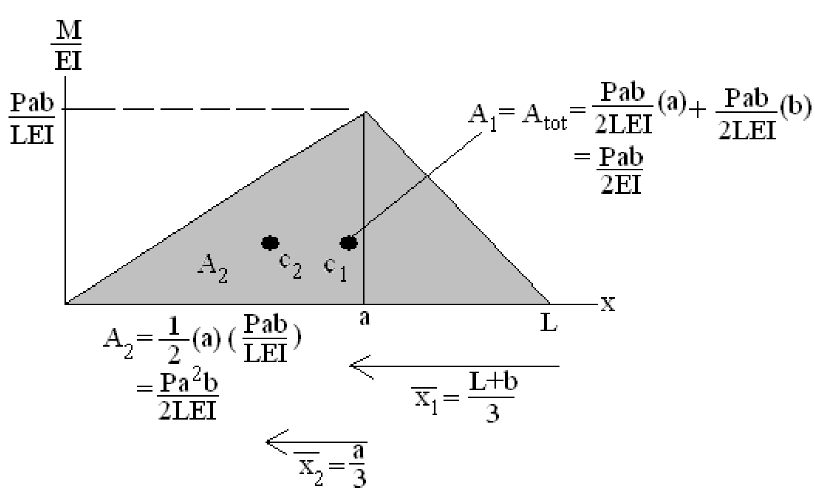
From similar triangles,

$$\frac{t_{B/A}}{L} = \frac{Pab}{6LEI}(L+b) = \frac{z}{b} \quad \left(\theta_A = \frac{t_{B/A}}{L}\right)$$

$$\Rightarrow z = \frac{Pab^2}{6LEI}(L+b)$$

$$t_{D/A} = A_2 \bar{x}_2 = \left(\frac{Pa^2b}{2LEI}\right)\left(\frac{a}{3}\right) = \frac{Pa^3b}{6LEI}$$

$$\delta_D = t_{B/A} - (z + t_{D/A}) = \boldsymbol{\frac{Pa^2b^2}{3LEI}}$$



note: In either of these last two examples, a general formula for δ would have been possible using the moment-area method.

Gere, James M. Mechanics of Materials: Sixth Edition. Brooks/Cole. Belmont, CA 2004.
Lee, Vincent. Lecturer. University of Southern California. CE225. Spring 2005.