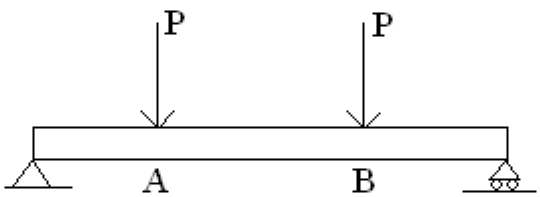
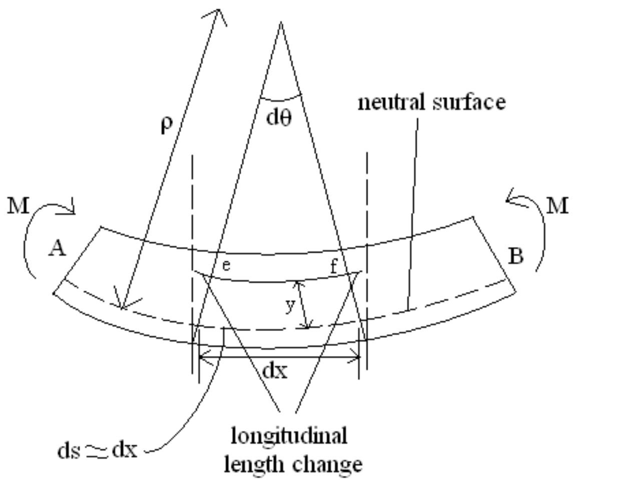
Bending



The following derivation will assume "pure bending" (bending moment is constant/shear force V = 0) and prismatic material. Longitudinal lines in the lower part of the beam are elongated (T) while those in the upper part are shortened (C). Somewhere between the top and bottom of the beam is a longitudinal surface in which there is no length change. This surface is the neutral surface. It passes through the centroid of the cross-sectional area, assuming that the cross sectional area is symmetrical about the xy plane and load resultants act in this plane.



ρ = radius of curvature

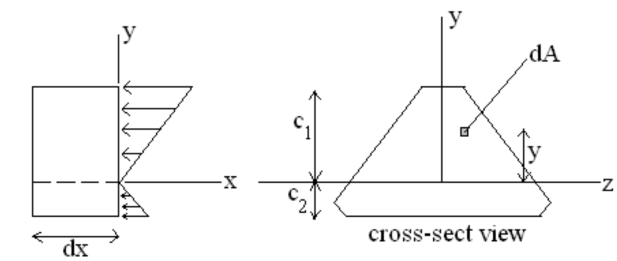
curvature = $\kappa = \dfrac{1}{\rho}$

If the flexure is small, ρ is large and κ is small.

$\dfrac{ds}{2\pi\rho}(2\pi) = d\theta$ where $\dfrac{ds}{2\pi\rho}$ is the fraction of arc length change, and 2π radians = 360°.

So, $\rho d\theta = ds \quad \kappa = \dfrac{d\theta}{ds}$ we deal with *very* small flexure, so $\kappa \approx \dfrac{d\theta}{dx}$ (θ in radians)

Now we're ready to find stress, strain, curvature, and deflection, in terms of bending moment.



note: for negative bending moment, the arrows are reversed (compression on bottom, tension on top)

Deflection (at midpoint) = $\delta = \rho - \rho\cos(\dfrac{d\theta}{2})$

$d\theta = \dfrac{ds}{\rho} \approx \dfrac{L}{\rho} \quad \delta = \rho - \rho\cos(\dfrac{L}{2\rho})$

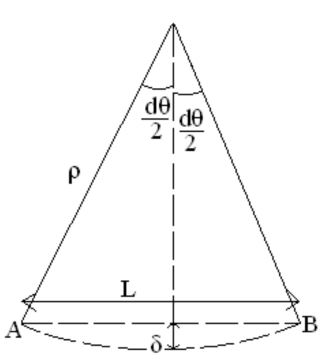
An arbitrary line ef above the x axis will shorten. Its original length = dx and its final length =

$(\rho - y)d\theta \approx (\rho - y)(\dfrac{dx}{\rho}) = dx - \dfrac{y}{\rho}dx$

longitudinal strain = $\dfrac{\text{longit length change}}{\text{original length}} = \varepsilon$

$= \dfrac{(dx - {}^{y}\!/_{\rho}\,dx) - dx}{dx} = \dfrac{-y}{\rho} \qquad \varepsilon = \dfrac{-y}{\rho} = -\kappa y$

$\sigma = E\varepsilon = \dfrac{-E\,y}{\rho} = -E\,\kappa\,y$



Need to find a relationship between σ or κ and M :

$M = \int_A (\frac{\text{force}}{\text{area}})(\text{dist})(\text{area}) = -\int_A \sigma\, y\, dA = \int_A \kappa\, E\, y^2\, dA$ If area moment of inertia = I =

$\int_A y^2 dA$, then $\kappa = \frac{M}{EI}$ $\quad \delta = \frac{EI}{M} - \frac{EI}{M}\cos(\frac{ML}{2EI})$ $\quad \varepsilon = \frac{-M}{EI} y$ note: $I \neq I_p$

$\sigma = -E(\frac{M}{EI})y = -\frac{My}{I}$ maximum tensile and compressive bending stresses occur at points located farthest from the neutral axis.

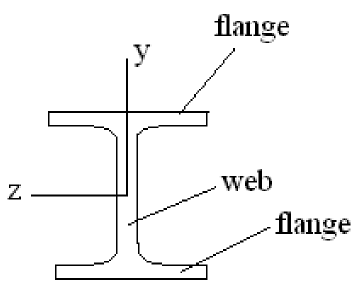
$$\mathbf{(\sigma_1)_{max} = \frac{-Mc_1}{I} \quad (\sigma_2)_{max} = \frac{Mc_2}{I}}$$

For positive M, σ_1 is compressive, σ_2 is tensile. For negative M, σ_1 is tensile, σ_2 is compressive. So, there are up to four strength conditions to check to determine σ_{max} for a given prismatic beam.

- see next example for center of mass and I calculation of an odd shape.

Rectangular cross-sect: $I = \frac{bh^3}{12}$

Circular cross-sect: $I = \frac{\pi d^4}{64}$



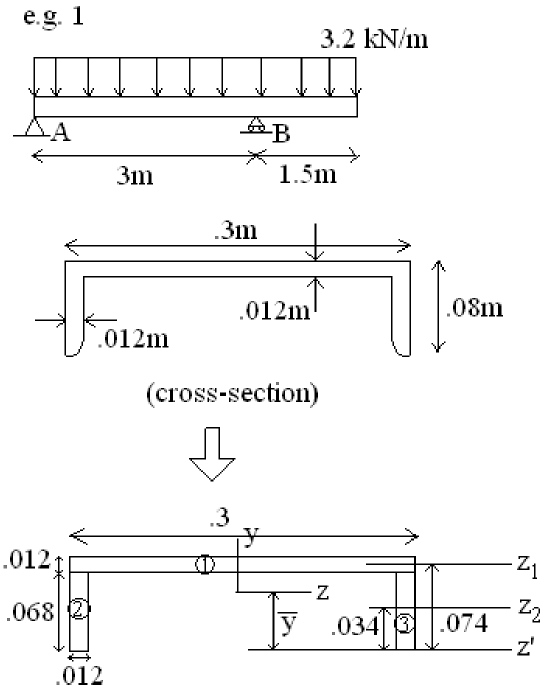
The “wide-flange” shape to the left approaches the ideal cross-sect shape for a beam of given area and height. The narrowness of the web is limited only by the shear stress.

note: small deflections only
note: these equations apply for cantilevered beams too
note: bending stress is NOT significantly altered by the presence of shear stresses, so

$\sigma = \frac{-My}{I}$ can be used for non-uniform bending with M_{max} yielding σ_{max}.

e.g. 1

Given: Beam with uniform cross-section shown and uniform load.



Find: $(\sigma_c)_{max}$ and $(\sigma_t)_{max}$.

Neutral axis from equivalent moments:

If $\bar{y} = c_2$ *, then* $(.3)(.012)(.074 - \bar{y}) =$
$2(.068)(.012)(\bar{y} - .034)$ *where (.3)(.012) is* A_1 *and 2(.068)(.012) is* $2A_2$.

$\Rightarrow \bar{y} = .06152m$

$\Rightarrow c_2 = .06152m$ *and* $c_1 = .08 - .06152m = .0185m$

General formula: $\bar{y} = \dfrac{\sum M_{z'}}{\sum A} = \dfrac{\sum A_i d_i}{\sum A_i}$

z' can be ANY parallel axis. d_i = *dist from* A_i *(* A_i *neutral axis) to z'.*

$\bar{y}$ = *dist from z' to z.*

$I_z = \sum [(I_i)_{zi} + A_i d_i^2]$ *where* d_i = *dist from* z_i *to z (see pic to the left)*

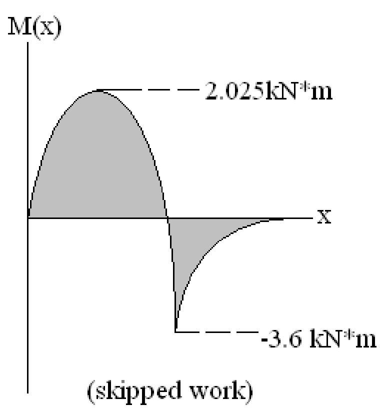
$$(I_1)_{z1} = \frac{1}{12}(.3)(.012)^3$$

$$(I_1)_z = \frac{1}{12}(.3)(.012)^3 + (.3)(.012)(.074 - .06152)^2 = 6.04x10^{-7}$$

$$(I_2)_z = (I_3)_z = \frac{1}{12}(.012)(.068)^3 + (.068)(.012)(.06152 - .034)^2 = 9.32x10^{-7}$$

$$I_z = \sum I = (6.04x10^{-7}) + 2(9.32x10^{-7}) = 2.47x10^{-6}\, m^4$$

note: the above formulas assume symmetry about y. I_z *formula also assumes symmetry about z. – i.e.* A_1 *is symmetrical about* z_1 *and* A_2 *and* A_3 *are symmetrical about* z_2.



Between AB: $(\sigma_1)_{max} = \dfrac{(2.025x10^3)(.0185)}{2.47x10^{-6}} = 15.2MPa\ (C)$

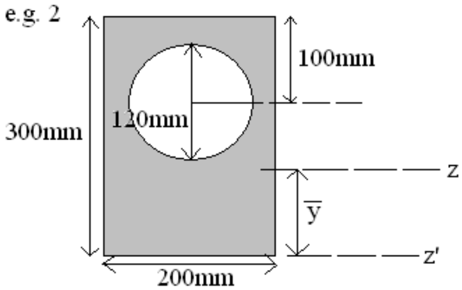
$$(\sigma_2)_{max} = \frac{(2.025x10^3)(.06152)}{2.47x10^{-6}} = 50.4MPa\ (T)$$

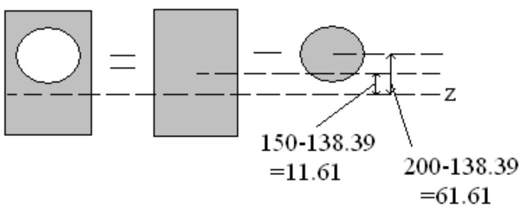
Between BC: $(\sigma_1)_{max} = \dfrac{(3.6x10^3)(.0185)}{2.47x10^{-6}} = 27.0MPa\ (T)$

$$(\sigma_2)_{max} = \frac{(3.6x10^3)(.06152)}{2.47x10^{-6}} = 89.7MPa\ (C)$$

$\Rightarrow (\sigma_c)_{max} =$ ***89.7 Mpa*** $(\sigma_t)_{max} =$ ***50.4 MPa***

e.g. 2
Given: Beam cross section.
Find: c_1, c_2, I





choose z' at bottom ($\bar{y} = c_2$):

$$\bar{y} = \frac{\sum A_i d_i}{\sum A_i} =$$

$$\frac{(200)(100)(50) + [(200)(200) - \pi/4(120)^2](200)}{(300)(200) - \pi/4(120)^2}$$

$= 138.39mm$

where (200)(100) is the area of the lower third and $(200)(200) - \pi/4(120)^2$ *is the area of the upper two-thirds.*

note: The upper two-thirds has height of 200mm. Looking at the dimensions on the figure, we can see that there is a height of 40mm above and 40mm below the cut-out circle within this upper two-thirds block. This symmetry explains why the last term in the numerator, d_i*, is 200 (goes from z' to the midpoint of the circle).*

OR

$$\bar{y} = \frac{(300)(200)(150) - \pi/4(120)^2(200)}{(300)(200) - \pi/4(120)^2} = 138.39mm$$

where (300)(200) is the area of the solid rectangle and $\pi/4(120)^2$ *is the area of the circle.*

$$(I_{rect})_z = \frac{1}{12}(200)(300)^3 + (300)(200)(11.61)^2$$

$$(I_{circ})_z = \frac{\pi}{64}(120)^4 + \pi/4(120)^2(61.61)^2$$

$$I_z = (I_{rect})_z - (I_{circ})_z = 4.908x10^8\ mm^4$$

<u>Bending stress design examples</u>

Bending stress is not related to area and depends on more than one dimension (usually). It turns out, not surprisingly, that for a rectangular section in bending, $A \to 0$ as $h \to \infty$

and b $\rightarrow$ 0. This is not surprising since we know that sections such as the wide-flange section, with the majority of material away from the neutral axis, are most cost effective. Design is best done using tables of common sections.

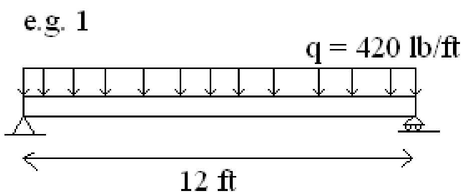
Often $\frac{I}{c_1}$ and $\frac{I}{c_2}$ are written as S_1 and S_2, where S = "section modulus." This simplifies the use of tables.

e.g. 1

Given: Wood beam with rectangular cross-sect, is subjected to the load shown. density

$$= 35\ \frac{lb}{ft^3} \quad \sigma_{allow} = 1800\,psi\,.$$

Find: Suitable size beam (base x height) from appendix A.

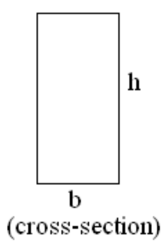


Simply supported beam (pin + roller), uniform load.

$$\Rightarrow M_{max} = \frac{qL^2}{8} = \frac{(420\,{}^{lb}\!/\!_{ft})(12\,ft)^2(12\,{}^{in}\!/\!_{ft})}{8}$$

*=90,720 lb*in (located at midpoint, skipped work)*

$$\sigma_{allow} = \frac{M_{max}}{S} \Rightarrow S = \frac{90{,}720}{1800} = 50.40in^3$$



From Appendix A, choose the lightest (smallest cross-sect area) beam that has a section modulus S of at least 50.40$\rightarrow$choose 3"x12". (3"x12" nominal dimensions, 2.5"x11.25" actual dimensions, s=52.73 in^3) But we still have to include the beam's own weight:

Beam weight = (area)(density)= $\left(\frac{2.5in}{12\,{}^{in}\!/\!_{ft}}\,x\,\frac{11.25in}{12\,{}^{in}\!/\!_{ft}}\right)(35\,{}^{lb}\!/\!_{ft^3}) = 6.8\,{}^{lb}\!/\!_{ft}$

$$S = \frac{90{,}720 + \frac{(6.8)(12)^2(12)}{8}}{1800} = 51.22in^3 \text{ or } S = (50.40)\left(\frac{426.8}{420}\right) = 51.22in^3$$

This is still smaller than the section modulus for the ***3 x 12 in.*** *beam, so that size is satisfactory.*

note: If $c_1 \neq c_2$, then the problem is more complicated, but still follows the same basic process.

note: We have ignored the phenomenon known as "lateral torsional buckling", which will be emphasized in the outline on steel design later on.

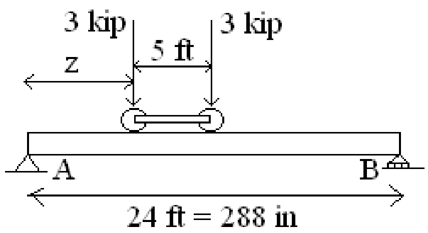
e.g. 2

Given: beam supports the two-wheeled vehicle shown.

It may occupy any position on the beam. $\sigma_{allow} = 21.4ksi$

Find: M_{max} *and the corresponding* S_{min}.

e.g. 2



In terms of arbitrary distance z from the left:
Support reactions:

$$\curvearrowleft + \sum M_A : B(288) - 3(z) - 3(z+60) = 0$$

$$\Rightarrow B = \frac{6z+180}{288} kip$$

$$+\uparrow \sum F_y : A + \frac{6z+180}{288} - 6 = 0 \Rightarrow A = \frac{1548-6z}{288} kip$$

$$x = 0^- : V(0^-) = M(0^-) = 0$$

$$x = 0^+ : V(0^+) = \frac{1548-6z}{288} \quad M(0^+) = 0$$

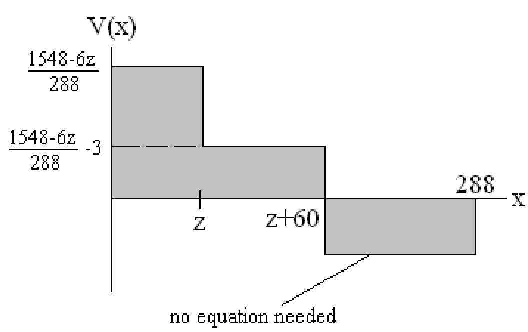
$$0^+ \le x \le z^- : V(x) = \frac{1548-6z}{288} \quad M(x) = \int_0^x \frac{1548-6z}{288} dx = \frac{1548-6z}{288} x$$

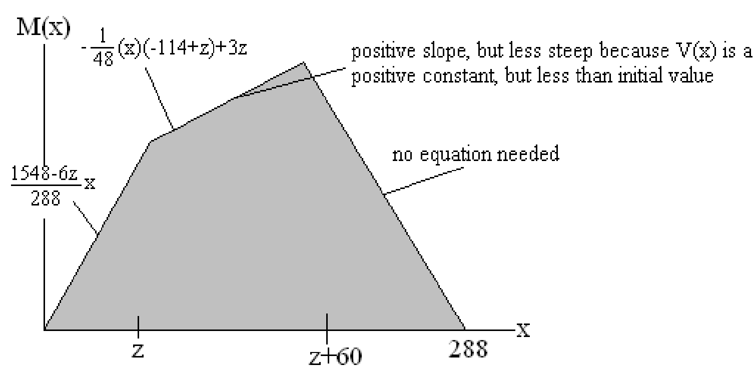
$$x = z^- : V(z^-) = \frac{1548-6z}{288} \quad M(z^-) = \frac{1548-6z}{288} z$$

$$x = z^+ : V(z^+) = \frac{1548-6z}{288} - 3 \quad M(z^+) = \frac{1548-6z}{288} z$$

$$z^+ \le x \le (z+60)^- : V(x) = \frac{1548-6z}{288} - 3 \quad M(x) = \frac{1548-6z}{288} z + \int_z^x \frac{1548-6z}{288} - 3dx$$

$$= -\frac{1}{48}(x)(-114+z) + 3z$$





$$M_{max} = M(z+60) = -\frac{1}{48}(z+60)(-114+z) + 3z \qquad z \; \varepsilon \; [0, 288-60]$$

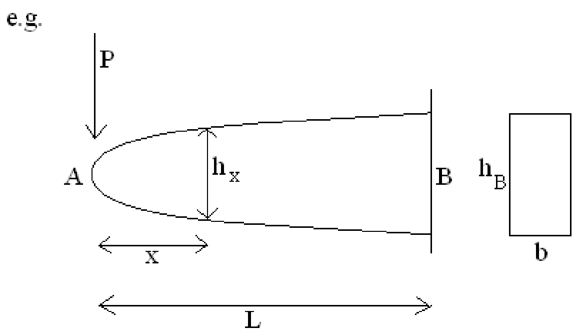
$$\Rightarrow \boldsymbol{z_{max} = 99in,\ M_{max} = 346.7 kip * in \quad S_{min} = \frac{346.7}{21.4} = 16.2in^3}$$

note: could then use a table to choose an efficient beam size.

<u>Tapered beams</u>

To really minimize the amount of material, the cross-sect dimensions can be varied so as to develop the maximum allowable bending stress at every section.

e.g.
Given: cantilevered beam with point load shown.
Find: h_x *so that* $\sigma = \sigma_{allow}$ *at every cross-section.*

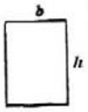


$$\sigma_{allow} = \frac{M_x\left(\frac{h_x}{2}\right)}{\left(\frac{1}{12}bh_x^3\right)} = \frac{6Px}{bh_x^2} \Rightarrow \boldsymbol{h_x = \sqrt{\frac{6Px}{b\sigma_{allow}}}}$$

note: If $c_1 \neq c_2$, *then the problem gets a bit more complicated.*
note: angle of taper must not be too large.

Gere, James M. <u>Mechanics of Materials: Sixth Edition</u>. Brooks/Cole. Belmont, CA 2004.
Lee, Vincent. Lecturer. University of Southern California. CE225. Spring 2005.

APPENDIX



Appendix A

SECTION PROPERTIES FOR SAWN LUMBER AND TIMBER

Nominal Size $b \times h$ in.	Standard Dressed Size (S4S) $b \times h$ in.	Area of Section A in.^{2}	X-X Axis		Y-Y Axis		Board Measure per Lineal Foot	Weight in pounds per linear foot of piece when weight of wood per cubic foot equals:		
			Moment of Inertia I in.^{4}	Section Modulus S in.^{3}	Moment of Inertia I in.^{4}	Section Modulus S in.^{3}		25 pcf	30 pcf	35 pcf
1 × 3	¾ × 2½	1.875	0.977	0.781	0.088	0.234	¼	0.326	0.391	0.456
1 × 4	¾ × 3½	2.625	2.680	1.531	0.123	0.328	⅓	0.456	0.547	0.638
1 × 6	¾ × 5½	4.125	10.398	3.781	0.193	0.516	½	0.716	0.859	1.003
1 × 8	¾ × 7¼	5.438	23.817	6.570	0.255	0.680	⅔	0.944	1.133	1.322
1 × 10	¾ × 9¼	6.938	49.466	10.695	0.325	0.867	⅚	1.204	1.445	1.686
1 × 12	¾ × 11¼	8.438	88.989	15.820	0.396	1.055	1	1.465	1.758	2.051
2 × 3*	1½ × 2½	3.750	1.953	1.563	0.703	0.938	½	0.651	0.781	0.911
2 × 4*	1½ × 3½	5.250	5.359	3.063	0.984	1.313	⅔	0.911	1.094	1.276
2 × 6*	1½ × 5½	8.250	20.797	7.563	1.547	2.063	1	1.432	1.719	2.005
2 × 8*	1½ × 7¼	10.875	47.635	13.141	2.039	2.719	1⅓	1.888	2.266	2.643
2 × 10*	1½ × 9¼	13.875	98.932	21.391	2.602	3.469	1⅔	2.409	2.891	3.372
2 × 12*	1½ × 11¼	16.875	177.979	31.641	3.164	4.219	2	2.930	3.516	4.102
2 × 14*	1½ × 13¼	19.875	290.775	43.891	3.727	4.969	2⅓	3.451	4.141	4.831
3 × 4	2½ × 3½	8.750	8.932	5.104	4.557	3.646	1	1.519	1.823	2.127
3 × 6	2½ × 5½	13.750	34.661	12.604	7.161	5.729	1½	2.387	2.865	3.342
3 × 8	2½ × 7¼	18.125	79.391	21.901	9.440	7.552	2	3.147	3.776	4.405
3 × 10	2½ × 9¼	23.125	164.886	35.651	12.044	9.635	2½	4.015	4.818	5.621
3 × 12	2½ × 11¼	28.125	296.631	52.734	14.648	11.719	3	4.883	5.859	6.836
3 × 14	2½ × 13¼	33.125	484.625	73.151	17.253	13.802	3½	5.751	6.901	8.051
3 × 16	2½ × 15¼	38.125	738.870	96.901	19.857	15.885	4	6.619	7.943	9.266
4 × 4	3½ × 3½	12.250	12.505	7.146	12.505	7.146	1⅓	2.127	2.552	2.977
4 × 6	3½ × 5½	19.250	48.526	17.646	19.651	11.229	2	3.342	4.010	4.679
4 × 8	3½ × 7¼	25.375	111.148	30.661	25.904	14.802	2⅔	4.405	5.286	6.168
4 × 10	3½ × 9¼	32.375	230.840	49.911	33.049	18.885	3⅓	5.621	6.745	7.869
4 × 12	3½ × 11¼	39.375	415.283	73.828	40.195	22.969	4	6.836	8.203	9.570
4 × 14	3½ × 13¼	46.375	678.475	102.411	47.340	27.052	4⅔	8.047	9.657	11.266
4 × 16	3½ × 15¼	53.375	1,034.418	135.661	54.487	31.135	5⅓	9.267	11.121	12.975
6 × 6	5½ × 5½	30.250	76.255	27.729	76.255	27.729	3	5.252	6.302	7.352
6 × 8	5½ × 7½	41.250	193.359	51.563	103.984	37.813	4	7.161	8.594	10.026
6 × 10	5½ × 9½	52.250	392.963	82.729	131.714	47.896	5	9.071	10.885	12.700
6 × 12	5½ × 11½	63.250	697.068	121.229	159.443	57.979	6	10.981	13.177	15.373
6 × 14	5½ × 13½	74.250	1,127.672	167.063	187.172	68.063	7	12.891	15.469	18.047
6 × 16	5½ × 15½	85.250	1,706.776	220.229	214.901	78.146	8	14.800	17.760	20.720
6 × 18	5½ × 17½	96.250	2,456.380	280.729	242.630	88.229	9	16.710	20.052	23.394
6 × 20	5½ × 19½	107.250	3,398.484	348.563	270.359	98.313	10	18.620	22.344	26.068
6 × 22	5½ × 21½	118.250	4,555.086	423.729	298.088	108.396	11	20.530	24.635	28.741
6 × 24	5½ × 23½	129.250	5,948.191	506.229	325.818	118.479	12	22.439	26.927	31.415
8 × 8	7½ × 7½	56.250	263.672	70.313	263.672	70.313	5⅓	9.766	11.719	13.672
8 × 10	7½ × 9½	71.250	535.859	112.813	333.984	89.063	6⅔	12.370	14.844	17.318
8 × 12	7½ × 11½	86.250	950.547	165.313	404.297	107.813	8	14.974	17.969	20.964
8 × 14	7½ × 13½	101.250	1,537.734	227.813	474.609	126.563	9⅓	17.578	21.094	24.609
8 × 16	7½ × 15½	116.250	2,327.422	300.313	544.922	143.313	10⅔	20.182	24.219	28.255
8 × 18	7½ × 17½	131.250	3,349.609	382.813	615.234	164.063	12	22.786	27.344	31.901
8 × 20	7½ × 19½	146.250	4,634.297	475.313	684.547	182.813	13⅓	25.391	30.469	35.547
8 × 22	7½ × 21½	161.250	6,211.484	577.813	755.859	201.563	14⅔	27.995	33.594	39.193
8 × 24	7½ × 23½	176.250	8,111.172	690.313	826.172	220.313	16	30.599	36.719	42.839
10 × 10	9½ × 9½	90.250	678.755	142.896	678.755	142.896	8⅓	15.668	18.802	21.936
10 × 12	9½ × 11½	109.250	1,204.026	209.396	821.651	172.979	10	18.967	22.760	26.554
10 × 14	9½ × 13½	128.250	1,947.797	288.563	964.547	203.063	11⅔	22.266	26.719	31.172
10 × 16	9½ × 15½	147.250	2,948.068	380.396	1,107.443	233.146	13⅓	25.564	30.677	35.790
10 × 18	9½ × 17½	166.250	4,242.836	484.896	1,250.338	263.229	15	28.863	34.635	40.408