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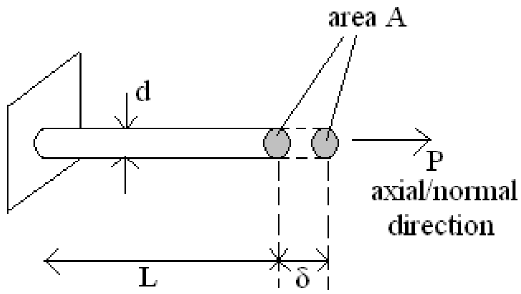
MECHANICS OF ELASTIC MATERIALS

"Mechanics of Materials" is typically an engineering student's first exposure to the important concepts relating to material properties, such as material strength and material stiffness. Material strength and stiffness are important for the analysis of structures, since the equations of static equilibrium are not enough to determine the distribution of forces within a complex structure. In addition, knowledge of material strength and stiffness is vital for the design of structures, where the size (and corresponding cost) of a component of a structure, such as a beam, depends on both its resistance to excessive deformation (primarily a function of stiffness), and its ability to resist damage (primarily a function of strength).

As we will see, beginning with this outline, two of the most important quantities in structural engineering are "stress" and "strain." The stress and strain of a material are often linearly-related – a discovery that dates back to 1678, when Hooke famously stated "ut tensio, sic vis," meaning, "as the extension, so the force." The larger this ratio, the more stiff the material, and the greater its resistance to deformation. Keeping deformations small is sometimes a constraint in the design of structures.

A constraint that is even more often present in engineering design is to ensure that the material strength, which has units of stress, is not exceeded. Stress demands, unlike strains, are not so easy to "see" or directly measure, but stress is a quantity that engineers like to use for the purpose of comparing to material strength. Designing a structure so that the stress demands in all of its structural components remain less than their corresponding material strength values is one way that an engineer can ensure that the structure is safe to perform its intended function.

Hooke's Law



Normal stress = $\dfrac{\text{force}}{\text{cross}-\text{sectional area}} = \dfrac{P}{A} = \sigma$

(stress distribution in units of $\dfrac{\text{lb}}{\text{in}^2}$, psi or $\dfrac{\text{kip}}{\text{in}^2}$, ksi or $\dfrac{\text{N}}{\text{m}^2}$, Pascal)

Normal strain =

$\dfrac{\text{elongation}}{\text{length}} = \dfrac{\delta}{L} = \varepsilon$ (fraction of change in length) (no units)

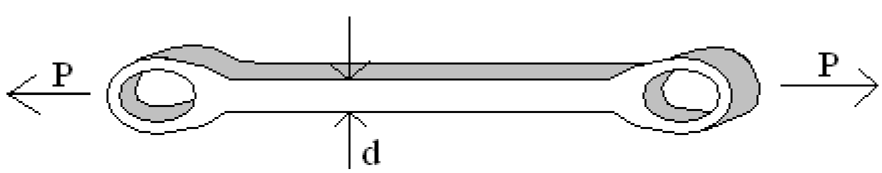
note: As the rod elongates, the area shrinks $\Rightarrow$ the actual stress is slightly larger than that assumed above. Similarly, the actual strain is actually $\dfrac{\delta}{L+\delta}$, which is slightly smaller than that assumed above.

Most general form of Hooke's Law:

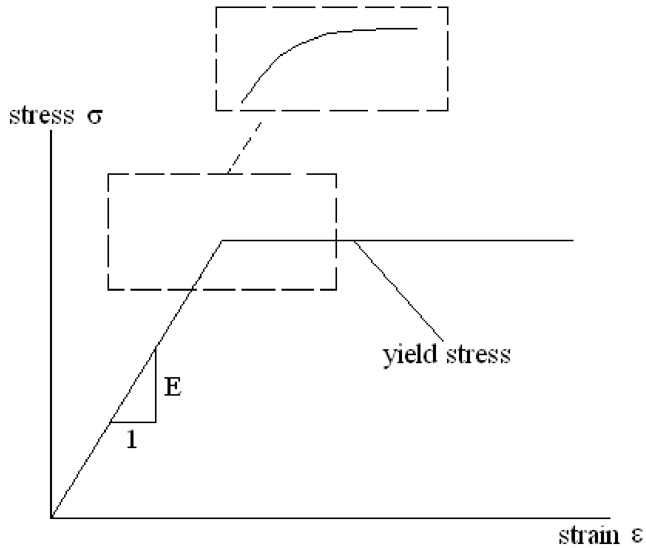
$\sigma = E\varepsilon$ where E = modulus of elasticity (material property)

note: Unless stated otherwise, P is assumed to be an equivalent force through the centroid of A.

note: As a practical rule, $\sigma =$ P/A may be used with good accuracy at any point within a bar that is at least as far away from the force concentration as the lateral dimension of that bar (d or greater in the picture below).



note: for non-uniform bars, such as the eyebar above, as long as you make sure that failure will occur in the prismatic portion of the beam, it can be analyzed using the normal stress and strain equations above.



The above picture is a graph of stress versus strain for typical structural steel. We can see that the slope of the curve is E, as one would expect according to the Hooke's Law equation stated above. Many materials obey this linear relationship. In addition, this portion is called the "elastic" portion, because any structure that is stressed within the elastic portion will return to its original state upon release of stress. The "yield stress" as shown on the graph is typically considered the limit of the material. Once the material reaches this stress, it continues to stretch or compress without any further load, and upon release of all load, will only partially return to its original state. This phenomenon is called "yielding." The yield stress value, which is a material property, is typically considered the *strength* of the material. Grade 50 steel, for example, has a yield stress of 50 ksi.

Often, materials are idealized as perfectly "elasto-plastic." As we can see on the diagram above, the steel is perfectly elastic, then perfectly plastic (yielding portion = "plastic" portion). The dotted box in the diagram, which shows the more a smooth transition, is sometimes neglected.

In this chapter, all materials will be assumed linear-elastic.

$$\delta = \frac{PL}{AE} \qquad AE = \text{"}axial\ rigidity\text{"}$$

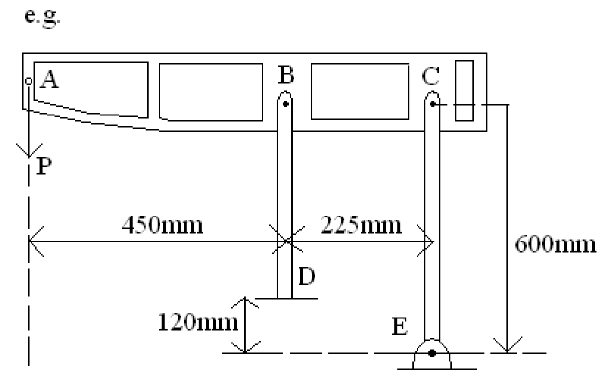
$$\delta = fP \qquad f = flexibility = \frac{L}{AE} \qquad \left(\frac{length\ change\ produced}{unit\ force}\right)$$

$$P = k\delta \quad k = stiffness = \frac{AE}{L} \quad \left(\frac{force\ required}{unit\ of\ length\ change}\right) \quad (commonly\ written\ version\ of\ Hooke's\ Law)$$

e.g.

Given: Dimensions of frame shown below. L_{BD} = *480mm*, L_{CE} = *600mm*, A_{BD} = *1020mm*^{2}, A_{CE} = *520mm*^{2}, E_{steel} = *205GPa*

Find: Assuming member ABC to be rigid, find P_{max} *if the displacement at point A is limited to 1.0mm.*



$$\circlearrowleft^{+} \sum M_B : P(450) - F_{CE}(225) = 0 \Rightarrow$$

F_{CE} = *2P*

$$+\uparrow \sum Fy : F_{BD} - 2P - P = 0 \Rightarrow F_{BD} = 3P$$

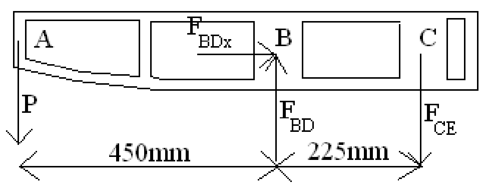
$$\xrightarrow{+} \sum Fx = 0 \Rightarrow F_{BDx} = 0$$

$$\delta_{BD} = \frac{(3P)(480x10^{-3})}{(205x10^{9})(1020x10^{-6})} = 6.887Px10^{-9}\ m$$

(shortening)

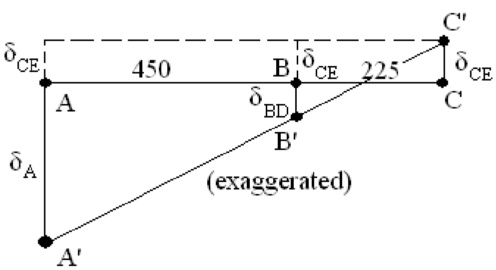
$$\delta_{CE} = \frac{(2P)(600x10^{-3})}{(205x10^{9})(520x10^{-6})} = 1.126Px10^{-8}\ m$$

(lengthening)



B moves to B', C moves to C', and A moves to A' by an amount δ_A.

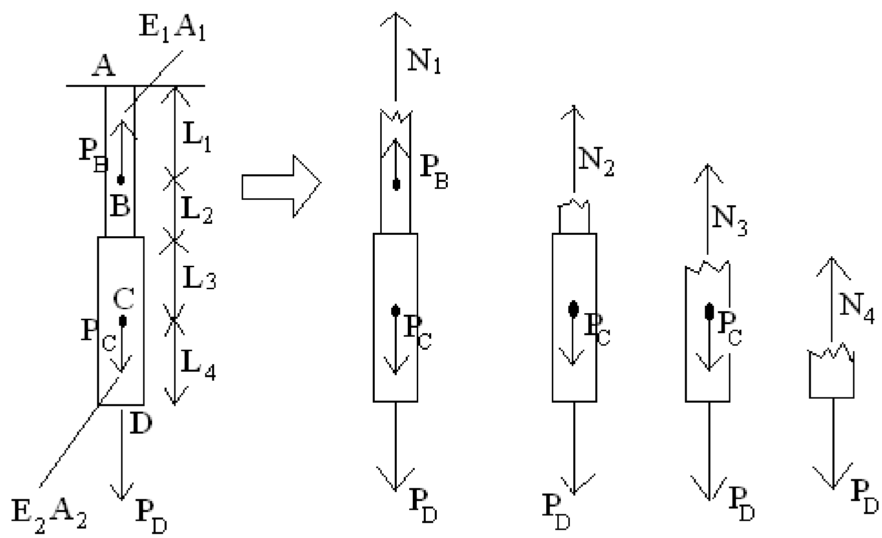
From similar triangles,



$$\frac{\delta_A + \delta_{CE}}{450 + 225} = \frac{\delta_{BD} + \delta_{CE}}{225}$$

$$\frac{(\delta_A)_{allowed} + (1.126P_{\max}x10^{-5}\,mm)}{450mm + 225mm} = \frac{(6.887P_{\max}x10^{-6}\,mm) + (1.126P_{\max}x10^{-5}\,mm)}{225mm}$$

substitute $(\delta_A)_{allowed} = 1.0mm$, *solve for* $P_{\max} \Rightarrow P_{\max} =$ ***23,200N***



$$\uparrow \sum Fy : N_1 + P_B - P_C - P_D = 0 \Rightarrow N_1 = P_C + P_D - P_B$$

$$N_2 = N_3 = Pc + P_D \qquad N_4 = P_D$$

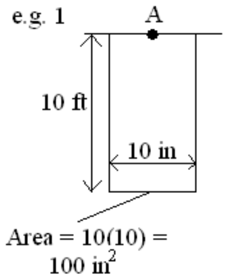
$$\delta_1 = \frac{N_1 L_1}{E_1 A_1} \quad \delta_2 = \frac{N_2 L_2}{E_1 A_1} \quad \delta_3 = \frac{N_3 L_3}{E_2 A_2} \quad \delta_4 = \frac{N_4 L_4}{E_2 A_2} \quad \delta_{total} = \sum \delta$$

<u>Deformation of tapered bars in tension</u>

Continuously varying loads or dimensions;

$$\mathrm{d}\delta = \frac{\mathrm{N(x)dx}}{\mathrm{EA(x)}} \qquad \delta = \int_0^L \frac{\mathrm{N(x)}}{\mathrm{EA(x)}}\mathrm{dx}$$

e.g. 1
Given: Square beam loaded by its own weight. Density = 10 kip/ft, E = 2000ksi (see pic below).
Find: δ



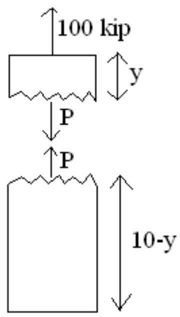
support reaction: $F_A = (10\frac{kip}{ft})(10\,ft) = 100kip$

Top piece:

$$+\uparrow \sum Fy : 100 - (10\frac{kip}{ft})(y) - P = 0 \Rightarrow P = 100 - 10y$$

Bottom piece:

$$+\uparrow \sum Fy : P - (10\frac{kip}{ft})(10 - y) = 0 \Rightarrow P = 100 - 10y \quad OK$$



$$\delta = \int_0^{10} \frac{100 - 10y}{(100in^2)(2000ksi)} dy = \frac{500kip * ft}{200{,}000kip} = \mathbf{.0025ft}$$

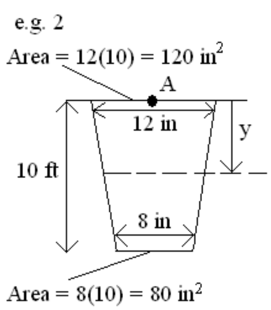
note: the general formula for length change of a bar (of constant A) subjected to uniform $\frac{force}{unit\ length}$ is $\delta = \frac{PL}{2EA}$ (P = 100kip in above problem)

e.g. 2

Given: Rectangular tapered beam of depth 10 in. loaded by its own weight. Same density and modulus of elasticity material as the above problem.

Find: δ

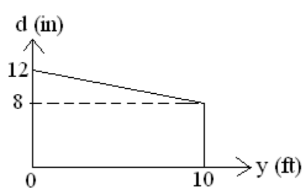
Density $\rho = 10\frac{kip}{ft}(\frac{1\,ft}{(12in)(100in^2)}) = \frac{1kip}{120in^3}$



Support reaction:

$$F_A = (\frac{120 + 80}{2} in^2)(10\,ft)(\frac{12in}{1\,ft})(\frac{1kip}{120in^3}) = 100kip$$

This used the fact that linearly changing area $\Rightarrow$
volume = (average area)(length)

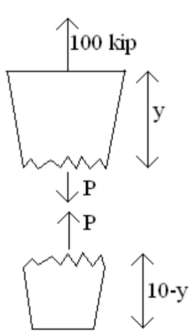


*A(y)=d(y)*depth=* $(\frac{12-8}{0-10}y + 12)(10)$ *=120-4y*

Top piece:

$$+\uparrow \sum Fy : 100 - P - (\frac{120 + (120 - 4y)}{2} in^2)(yft) * (\frac{12in}{1\,ft})(\frac{1kip}{120in^3}) = 0$$

$$\Rightarrow P = \frac{1}{5}(y^2 - 60y + 500)$$



$$\delta = \int_0^{10} \frac{\frac{1}{5}(y^2 - 60y + 500)}{(120 - 4y)(2000)} dy = \mathbf{.0022ft}$$

note: The tapered bar has slightly less elongation than a prismatic bar of equal length and volume.

note: The area must be constant or vary linearly or the problem is more complex. i.e. might be told that the top length of the tapered bar is 12 in and the bottom length is 8 in, and it is a circular cylinder. So, A = 36π and 16π respectively.

$A(y) = \frac{\pi}{4}[d(y)]^2 = \frac{\pi}{4}(\frac{12-8}{0-10}y+12)^2$ which is NOT linear. So, finding needed volumes is more complicated.

Simple statically indeterminate system (axial)

The following problems will be our first look at statically indeterminate (redundant) systems, as described in the section titled "A note on redundant systems" in the outline on Statics. By knowing the properties of the materials and Hooke's Law, which essentially relates force and displacement, we now have an additional equation to use for the purpose of finding unknown forces. This is called an equation of compatibility. All we have to do is find a way to relate displacements in members where we have unknown force(s).

e.g. 1

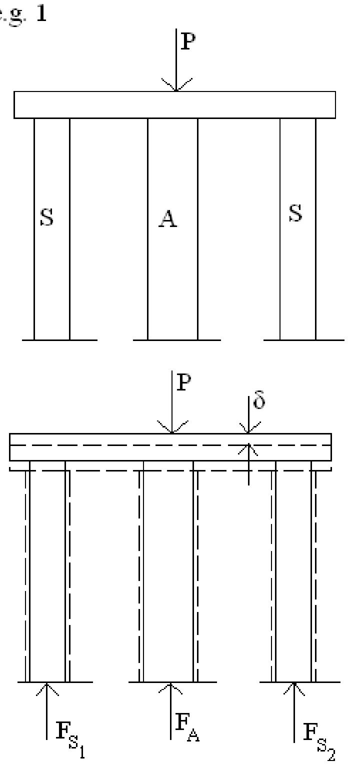
Given: Rigid bar of negligible weight rests on top of aluminum and steel beams. Force P acts at the midpoint.

Aluminum beam: diameter d_A = 1m, $(\sigma_A)_{allowed} = 80MPa, E_A = 70GPa$

Steel beams: diameter d_S = .5m, $(\sigma_S)_{allowed} = 220MPa$, $E_S = 210GPa$

Find: P_{max}

e.g. 1



From symmetry (or $\sum M_A = 0$),

$F_{S1} = F_{S2} = F_S$

$+\uparrow \sum Fy : F_A + 2F_S - P = 0$

Equation of compatibility:

Since the top bar is rigid,

$\delta_S = \delta_A$

$$\frac{F_S L}{E_S A_S} = \frac{F_A L}{E_A A_A} \quad (1)$$

From $\sum Fy = 0$, $F_A = P_{max} - 2F_S$ *(2)*

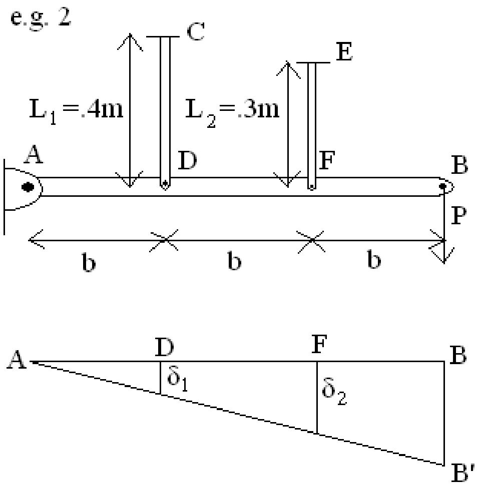
$(\sigma_S)_{allowed} = \frac{F_S}{A_S}$ *(3) and check* $(\sigma_A)_{allowed} \geq \frac{F_A}{A_A}$

OR

$(\sigma_A)_{allowed} = \frac{F_A}{A_A}$ *(3) and check* $(\sigma_S)_{allowed} \geq \frac{F_S}{A_S}$

3 equations, 3 unknowns $F_A, F_S, P_{\max} \rightarrow$ P_{max} *=* ***144 MN (at which point steel yields)***

e.g. 2
Given: Force P acts at the end of a rigid, pinned bar.
wire 1: $d_1 = 4mm, (\sigma_1)_{allowed} = 200MPa, E_1 = 72GPa$
wire 2: $d_2 = 3mm, (\sigma_2)_{allowed} = 175MPa, E_2 = 45GPa$
Find: P_{max}



$$\curvearrowleft + \sum M_A : T_1 b + T_2(2b) - P(3b) = 0$$

$$+\uparrow \sum Fy : T_1 + T_2 - Ay - P = 0$$

Equation of compatibility:

From similar triangles,

$$\frac{\delta_2}{2b} = \frac{\delta_1}{b} \Rightarrow \delta_2 = 2\delta_1$$

$$\frac{T_2 L_2}{E_2 A_2} = \frac{2T_1 L_1}{E_1 A_1} \quad (1)$$

From $\sum M_A = 0,\ T_1 = 3P_{\max} - 2T_2$ *(2)*
From $\sum Fy = 0,\quad Ay = T_1 + T_2 - P$ *(3)*

$(\sigma_1)_{allowed} = \dfrac{T_1}{A_1}$ *(4) and check:* $(\sigma_2)_{allowed} \geq \dfrac{T_2}{A_2}$

OR

$(\sigma_2)_{allowed} = \dfrac{T_2}{A_2}$ *(4) and check:* $(\sigma_1)_{allowed} \geq \dfrac{T_1}{A_1}$

4 equations, 4 unknowns $T_1, T_2, Ay, P_{\max} \rightarrow P_{\max} =$***1.26kN (at which point wire 2 yields)***

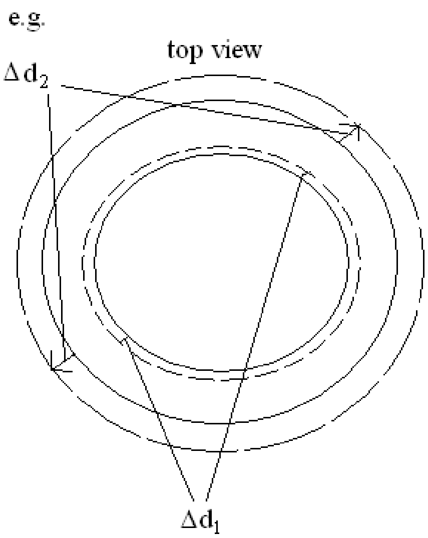
Poisson's Ratio

$$\text{Lateral strain} = \frac{\text{change in lateral length}}{\text{initial lateral length}} = \varepsilon' \text{ (no units)}$$

$\varepsilon' = \nu\varepsilon$ where ν = Poisson's ratio (material property) and recall the definition of ε from the beginning of this chapter (change in lateral length)=-(initial lateral length)(ν)(ε)

note: only applies to isotropic materials (same elastic properties in axial, lateral, or any direction). Concrete and most metals are isotropic. Wood is an example of an anisotropic (non-isotropic) material (it is much tougher against the grain).

e.g.
Given: Hollow polymer pipe of length 4 ft, outside diameter d_2 = 6 in., inside diameter d_1 = 4.5 in., is compressed by 140 kip normal force. E = 3000 ksi, ν = .3
Find: Increase in wall thickness Δt.



$$\Delta d_1 = d_1 \nu (\frac{P}{AE}) = 4.5(.3)(\frac{140}{\pi/4 (6^2 - 4.5^2)(3000)}) = .00509 in$$

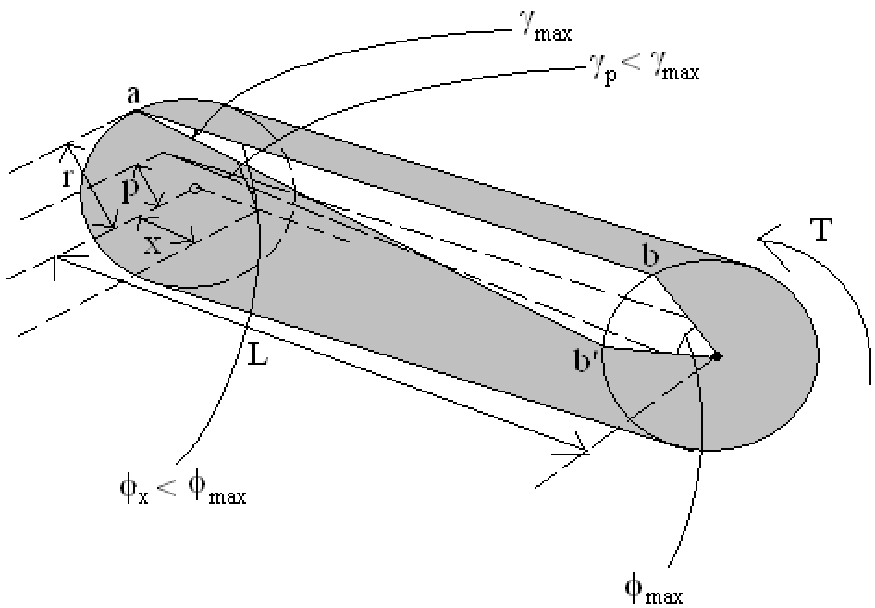
$$\Delta d_2 = d_2 \nu (\frac{P}{AE}) = 6(.3)(\frac{140}{\pi/4 (6^2 - 4.5^2)(3000)}) = .00679 in$$

$$\Delta t = \Delta r_2 - \Delta r_1 = \frac{\Delta d_2 - \Delta d_1}{2} = \boldsymbol{.00085 in}$$

note: under compression, outer diameter, inner diameter, and thickness all increase.

note: follow the same process for the lateral elongation (or shortening) for *each* dimension of a rectangular bar.

Torsion



Above is a fixed, prismatic beam subjected to a torque at the right end.

ϕ_{max} = angle of twist

ϕ_x depends on distance x from the wall

γ_p depends on distance p from the center

Assume distance bb' is very small and so the arc length bb' is approximately equal to a straight line bb'.

$$\phi_{max} = [\text{fraction of arc length change}](2\pi \text{ radians}) = [\frac{bb'}{2\pi\, r}](2\pi) = \frac{bb'}{r}$$

$$\gamma_{max} = [\frac{bb'}{2\pi(ab)}](2\pi) = \frac{bb'}{ab}$$

We can see that $\gamma_{max} = \frac{r\phi_{max}}{L}$ is really the same expression as above.

Also, $\gamma_p = \frac{p}{r}\gamma_{max} = p\frac{\phi_{max}}{L}$

$\tau = G\gamma$

$\tau_{max} = G\gamma_{max} = Gr\frac{\phi_{max}}{L} \qquad \tau_p = \frac{p}{r}\tau_{max} = Gp\frac{\phi_{max}}{L}$

We need to find a relationship between τ and T:

$$T = \int_A \tau_p p\, dA = \int_A (\frac{\tau_{max}}{r} p) p\, dA$$

If polar moment of inertia = $I_p = \int_A p^2 dA$, then:

$T = \frac{\tau_{max}}{r} I_p \Rightarrow \tau_{max} = \frac{Tr}{I_p} \rightarrow$ general formula for a circular shaft subjected to torsion

$$\boldsymbol{\phi_{max} = \frac{\tau_{max} L}{Gr} = \frac{TL}{GI_p} \qquad \gamma_{max} = \frac{r\phi_{max}}{L} = \frac{\tau_{max}}{G} = \frac{Tr}{GI_p}}$$

ϕ_{max} is often just written ϕ

note: $\phi_x = \frac{x}{L}\phi_{max} = \frac{\tau_{max} x}{Gr}$ (also note similarity of ϕ_{max} above to $\delta = \frac{PL}{EA}$)

Solid Bar:

$$T = \frac{\tau_{max}}{r} \int_{\phi=0}^{2\pi} \int_{p=0}^{r} p^2 p \, dp \, d\theta = (\frac{\tau_{max}}{r}) \frac{\pi r^4}{2} = \frac{\tau_{max} \pi d^3}{16} \Rightarrow$$

$$\boldsymbol{\tau_{max} = \frac{16T}{\pi d^3}} \textbf{ (solid shaft)}$$

note: recall from calculus that the extra p in the integrand is just an extra polar integration factor

Hollow Tube:

$$T = \frac{\tau_{max}}{r} \int_{\theta=0}^{2\pi} \int_{p=r1}^{r2} p^2 p \, dp \, d\theta = (\frac{\tau_{max}}{r}) \frac{\pi}{2} (r_2^4 - r_1^4) = (\frac{\tau_{max}}{2}) \frac{\pi}{32} (d_2^4 - d_1^4) \Rightarrow$$

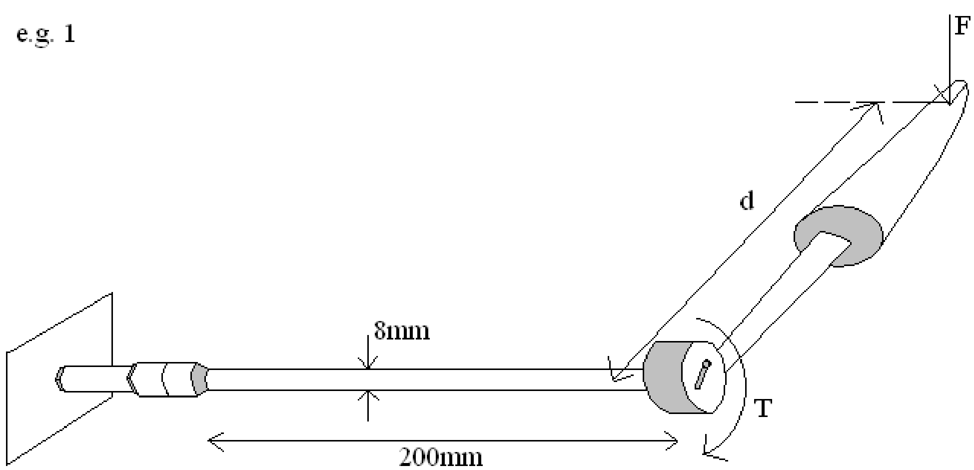
$$\boldsymbol{\tau_{max} = \frac{16Td_2}{\pi(d_2^4 - d_1^4)}} \quad \textbf{(tube)}$$

e.g. 1

Given: Socket wrench transmits torque to a stuck bolt.

$\tau_{allowable} = 460MPa \quad G = 78GPa$ *for the 8mm diameter, solid shaft shown*

Find: T_{max} *and* ϕ_{max} *for this allowable torque value*



$$\tau_{max} = \frac{16T}{\pi d^3} \quad T_{max} = \frac{(\tau_{allowable})\pi(8x10^{-3})^3}{16} = \mathbf{46.25N*m} \qquad (F_{max} = \frac{T_{max}}{d})$$

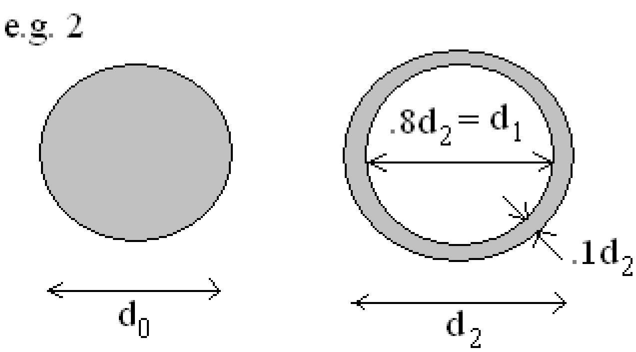
$$\phi = \frac{\tau_{max} L}{Gr} = \frac{(\tau_{allowable})(200x10^{-3})}{(78x10^9)(\frac{8}{2}x10^{-3})} = .29\,Rad \quad or\ (.29\,Rad)(\frac{180°}{\pi\,Rad}) = \mathbf{16.6°}$$

e.g. 2

Given: Either a solid or a hollow steel shaft is to be manufactured,

$T_{max} = 1200N*m \quad \tau_{allowable} = 40MPa \quad$ *Thickness of hollow shaft* $=.1\,d_2$

Find: $(d_0)_{min}, (d_2)_{min}$*, and the ratio of material usage for the hollow shaft versus the solid shaft.*



Solid:

$$\tau_{max} = \frac{16T}{\pi d^3} \Rightarrow (d_0)_{min} = \sqrt[3]{\frac{16T_{max}}{\pi\,\tau_{allowable}}} = \mathbf{53.5mm}$$

Tube:

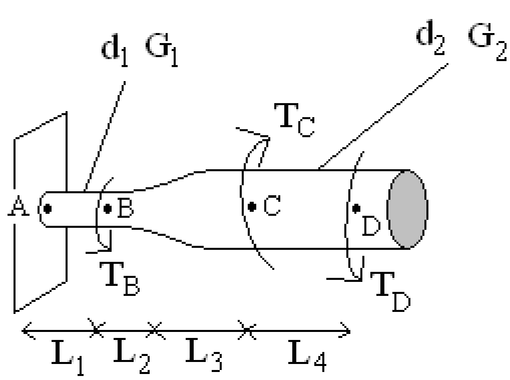
$$\tau_{max} = \frac{16Td_2}{\pi[d_2^{\,4} - (.8d_2)^4]} \Rightarrow$$

$$(d_2)_{min} = \sqrt[3]{\frac{16T_{max}}{(\tau_{allowable})\pi(1-.8^4)}} = \mathbf{63.7mm}$$

Since both shafts are the same density and length, the ratio of weights = the ratio of volumes = the ratio of areas:

$$\frac{A_{hollow}}{A_{solid}} = \frac{\pi/4\,(d_2^{\,2} - d_1^{\,2})}{\pi/4\, d_0^{\,2}} = \mathbf{.47}$$

The hollow shaft has a larger diameter, but only uses 47% as much material as the solid shaft. Hollow shafts are more efficient.



$$\phi_1 = \frac{T_1L_1}{G_1I_{p1}} \quad \phi_2 = \frac{T_2L_2}{G_1I_{p1}} \quad \phi_3 = \frac{T_3L_3}{G_2I_{p2}} \quad \phi_4 = \frac{T_4L_4}{G_2I_{p2}}$$

$$\phi_{total} = \sum\phi$$

T_1, T_2, T_3, T_4 *are the internal torques within sections* L_1, L_2, L_3, L_4*, respectively, which can be found from drawing free body diagrams as was done for the axial case in the previous section on Hooke's Law.*

<u>Deformation of tapered bars in torsion</u>

Continuously varying torques/dimensions;

$$d\phi = \frac{T(x)\,d(x)}{GI_p(x)} \qquad \phi = \int_0^L \frac{T(x)}{GI_p(x)}dx$$

note: satisfactory as long as angle of taper is less than 10°

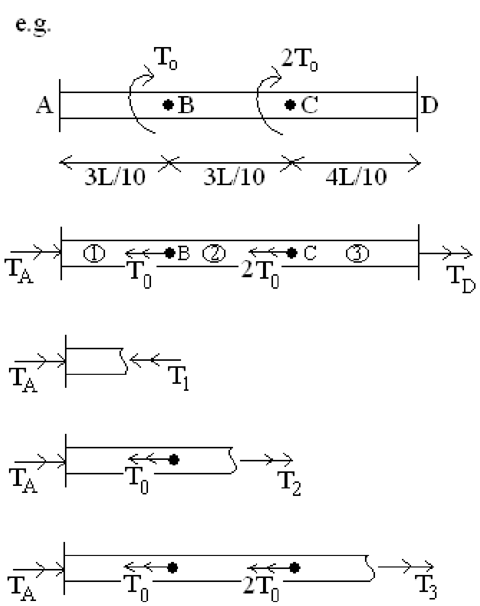
$I_p(x)$ determined from d(x), where d is the diameter

note: A shaft in torsion has a normal stress. If $\sigma_{allowable}$ for a material is equal to, or less than $\tau_{allowable}$, then <u>the design for the shaft in torsion is controlled by σ</u>. And, it will fail along a 45° axis. (proof – section on Mohr's Circle later in this chapter) (e.g. chalk)

note: Similarly, a member under axial load has a shear stress. If $\tau_{allowable}$ for a material is equal to, or less than $\frac{1}{2}(\sigma_{allowable})$, then <u>the design for the axially loaded member is controlled by τ</u>. And, it will fail along a 45° axis. (proof- section on Mohr's Circle) (e.g. concrete)

Simple statically indeterminate system (torsion)

e.g.
Given: Circular bar with fixed (rigid) ends shown.
Find: Support reactions and ϕ_{max}.



$$\overset{+}{\rightarrow}\ \sum T_A + T_D - T_0 - 2T_0 = 0 \ \ \textit{(eq of equilibrium)}$$

From the three FBD's to the left, we can see that:

$T_1 = T_A \quad T_2 = T_0 - T_A \quad T_3 = 3T_0 - T_A$

So,

$$\phi_1 = \frac{(T_A)\,{}^{3L}\!/_{10}}{GI_p} \qquad \phi_2 = \frac{(T_0 - T_A)\,{}^{3L}\!/_{10}}{GI_p}$$

$$\phi_3 = \frac{(3T_0 - T_A)\,{}^{4L}\!/_{10}}{GI_p}$$

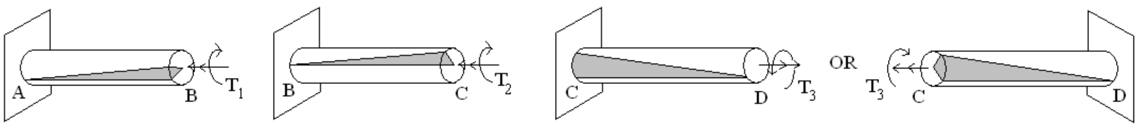
$$\overset{\rightarrow\!\rightarrow}{+}\sum \frac{-(T_A)\,{}^{3L}\!/_{10}}{GI_p} + \frac{(T_0 - T_A)\,{}^{3L}\!/_{10}}{GI_p} + \frac{(3T_0 - T_A)\,{}^{4L}\!/_{10}}{GI_p} = 0$$

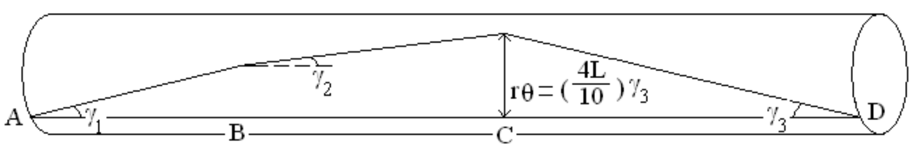
(eq of compatibility)

2 eq, 2 unknowns $\Rightarrow \boldsymbol{T_A = {}^{3T_0}\!/_{2} \quad T_D = 3T_0 - \frac{3T_0}{2} = {}^{3T_0}\!/_{2}}$

$$\phi_1 = \frac{({}^{3T_0}\!/_{2})\,{}^{3L}\!/_{10}}{GI_p} = \frac{9T_0L}{20GI_p} \qquad \phi_2 = \frac{(T_0 - {}^{3T_0}\!/_{2})\,{}^{3L}\!/_{10}}{GI_p} = \frac{-3T_0L}{20GI_p} \Rightarrow \phi_2 = \frac{3T_0L}{20GI_p} \leftarrow\!\leftarrow$$

$$\boldsymbol{\phi_3 = \frac{(3T_0 - {}^{3T_0}\!/_{2})\,{}^{4L}\!/_{10}}{GI_p} = \frac{3T_0L}{5GI_p} = \phi_{max}}$$





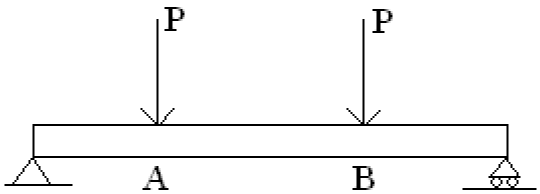
Quick summation of arc-lengths $r\gamma$ prove that

$$({}^{3L}\!/_{10})\gamma_1 + ({}^{3L}\!/_{10})\gamma_2 = ({}^{4L}\!/_{10})\gamma_3$$

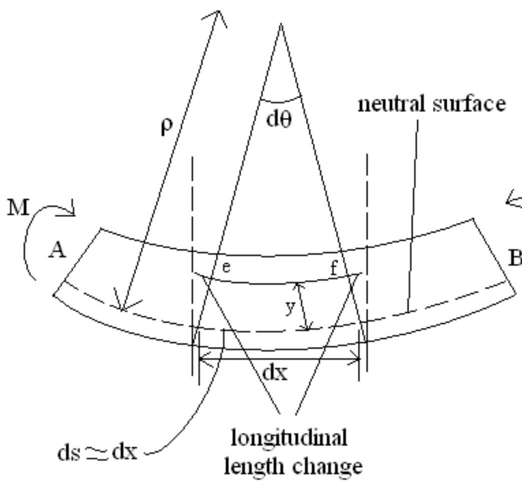
note: Direction of ϕ for each segment should be consistent with the direction of torques on free-body diagrams for each segment. Incorrect guess will simply result in negative values for ϕ.

note: As always with design, the allowable stress (τ_{allow}) and the force (T) are known, and we want to minimize the area A. Axial, bearing, and direct shear stresses are related to A, so we can easily minimize A. Shear stress for a solid shaft in torsion is not related to A, but it is related to d, so we can easily minimize A. The stress for a hollow tube in torsion, however, is not related to A and it depends on more than one dimension (d_1 and d_2). There are thus three unknowns (d_1, d_2, and A) and two equations ($A = \frac{\pi}{4}(d_2^{\,2} - d_1^{\,2})$, $\tau_{allow} = \frac{16Td_2}{\pi(d_2^{\,4} - d_1^{\,4})}$). One might be tempted to use $\frac{dA}{dx} = 0$ for a third equation, but there is no local minimum. It turns out, not surprisingly, that $A \to 0$ as $d_2 \to \infty$ and $d_1 \to d_2$. The best method, for this particular case, would be to use a table of common tube sizes and pick the tube with the smallest area in which $\tau \le \tau_{allow}$. In engineering practice, methods that utilize tables are often used, particularly for the selection of timber and steel section sizes for flexure, which we will learn about next.

Bending



The following derivation will assume “pure bending” (bending moment is constant/shear force V = 0) and prismatic material. Longitudinal lines in the lower part of the beam are elongated (T) while those in the upper part are shortened (C). Somewhere between the top and bottom of the beam is a longitudinal surface in which there is no length change. This surface is the neutral surface. It passes through the centroid of the cross-sectional area, assuming that the cross sectional area is symmetrical about the xy plane and load resultants act in this plane.



ρ = radius of curvature

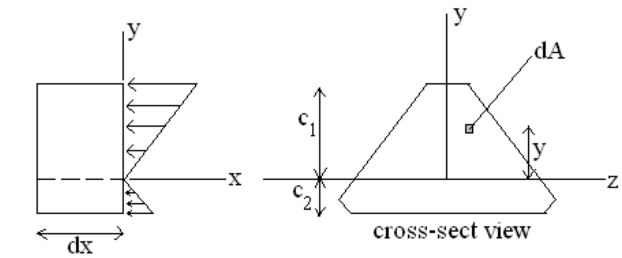
curvature = $\kappa = \dfrac{1}{\rho}$

If the flexure is small, ρ is large and κ is small.

$\dfrac{ds}{2\pi\rho}(2\pi) = d\theta$ where $\dfrac{ds}{2\pi\rho}$ is the fraction of arc length change, and 2π radians = 360°.

So, $\rho d\theta = ds \quad \kappa = \dfrac{d\theta}{ds}$ we deal with *very* small flexure, so $\kappa \approx \dfrac{d\theta}{dx}$ (θ in radians)

Now we’re ready to find stress, strain, curvature, and deflection, in terms of bending moment.



note: for negative bending moment, the arrows are reversed (compression on bottom, tension on top)

Deflection (at midpoint) = $\delta = \rho - \rho\cos(\dfrac{d\theta}{2})$

$d\theta = \dfrac{ds}{\rho} \approx \dfrac{L}{\rho} \qquad \delta = \rho - \rho\cos(\dfrac{L}{2\rho})$

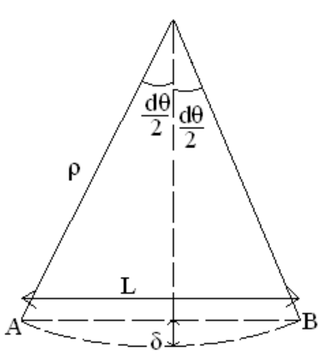
An arbitrary line ef above the x axis will shorten. Its original length = dx and its final length =

$(\rho - y)d\theta \approx (\rho - y)(\dfrac{dx}{\rho}) = dx - \dfrac{y}{\rho}dx$

longitudinal strain = $\dfrac{\text{longit length change}}{\text{original length}} = \varepsilon$

$= \dfrac{(dx - {}^{y}\!/_{\rho}\, dx) - dx}{dx} = \dfrac{-y}{\rho} \qquad \varepsilon = \dfrac{-y}{\rho} = -\kappa y$

$\sigma = E\varepsilon = \dfrac{-E\,y}{\rho} = -E\,\kappa\, y$



Need to find a relationship between σ or κ and M :

$M = \int_A (\frac{\text{force}}{\text{area}})(\text{dist})(\text{area}) = -\int_A \sigma\, y\, dA = \int_A \kappa E y^2\, dA$ If area moment of inertia = I = $\int_A y^2 dA$, then $\kappa = \frac{M}{EI}$ $\quad \delta = \frac{EI}{M} - \frac{EI}{M}\cos(\frac{ML}{2EI})$ $\quad \varepsilon = \frac{-M}{EI} y$ note: $I \neq I_p$

$\sigma = -E(\frac{M}{EI})y = -\frac{My}{I}$ maximum tensile and compressive bending stresses occur at points located farthest from the neutral axis.

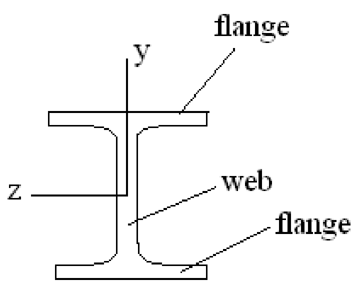
$$\boldsymbol{(\sigma_1)_{max} = \frac{-Mc_1}{I} \quad (\sigma_2)_{max} = \frac{Mc_2}{I}}$$

For positive M, σ_1 is compressive, σ_2 is tensile. For negative M, σ_1 is tensile, σ_2 is compressive. So, there are up to four strength conditions to check to determine σ_{max} for a given prismatic beam.

- see next example for center of mass and I calculation of an odd shape.

Rectangular cross-sect: $I = \frac{bh^3}{12}$

Circular cross-sect: $I = \frac{\pi d^4}{64}$

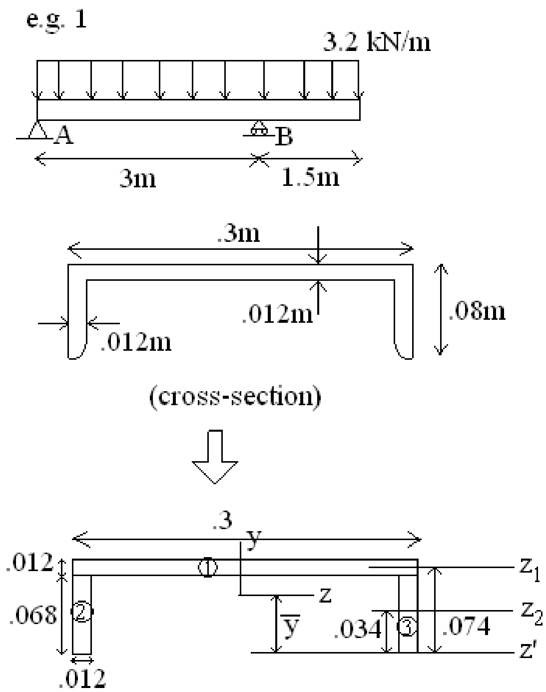


The "wide-flange" shape to the left approaches the ideal cross-sect shape for a beam of given area and height. The narrowness of the web is limited only by the shear stress.

note: small deflections only
note: these equations apply for cantilevered beams too
note: bending stress is NOT significantly altered by the presence of shear stresses, so $\sigma = \frac{-My}{I}$ can be used for non-uniform bending with M_{max} yielding σ_{max}.

e.g. 1

Given: Beam with uniform cross-section shown and uniform load.



Find: $(\sigma_c)_{max}$ and $(\sigma_t)_{max}$.

Neutral axis from equivalent moments:

If $\bar{y} = c_2$*, then* $(.3)(.012)(.074-\bar{y}) =$

$2(.068)(.012)(\bar{y}-.034)$ *where (.3)(.012) is* A_1 *and 2(.068)(.012) is* $2A_2$.

$\Rightarrow \bar{y} = .06152m$

$\Rightarrow c_2 = .06152m$ *and* $c_1 = .08 - .06152m = .0185m$

General formula: $\bar{y} = \dfrac{\sum M_{z'}}{\sum A} = \dfrac{\sum A_i d_i}{\sum A_i}$

z' *can be ANY parallel axis.* $d_i =$ *dist from* A_i *(*A_i *neutral axis) to* z'.

$\bar{y} =$ *dist from z' to z.*

$I_z = \sum[(I_i)_{zi} + A_i d_i^2]$ *where* $d_i =$ *dist from* z_i *to z (see pic to the left)*

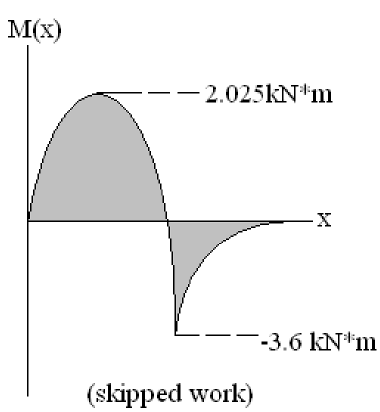
$$(I_1)_{z1} = \frac{1}{12}(.3)(.012)^3$$

$$(I_1)_z = \frac{1}{12}(.3)(.012)^3 + (.3)(.012)(.074-.06152)^2 = 6.04x10^{-7}$$

$$(I_2)_z = (I_3)_z = \frac{1}{12}(.012)(.068)^3 + (.068)(.012)(.06152-.034)^2 = 9.32x10^{-7}$$

$$I_z = \sum I = (6.04x10^{-7}) + 2(9.32x10^{-7}) = 2.47x10^{-6}\,m^4$$

note: the above formulas assume symmetry about y. I_z *formula also assumes symmetry about z. – i.e.* A_1 *is symmetrical about* z_1 *and* A_2 *and* A_3 *are symmetrical about* z_2.



Between AB: $(\sigma_1)_{max} = \dfrac{(2.025x10^3)(.0185)}{2.47x10^{-6}} = 15.2MPa\ (C)$

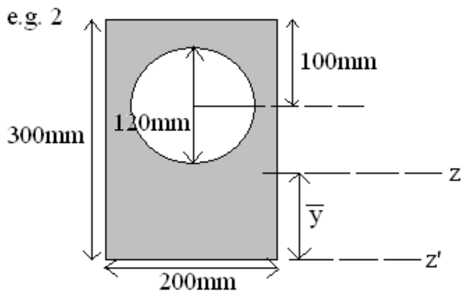
$$(\sigma_2)_{max} = \frac{(2.025x10^3)(.06152)}{2.47x10^{-6}} = 50.4MPa\ (T)$$

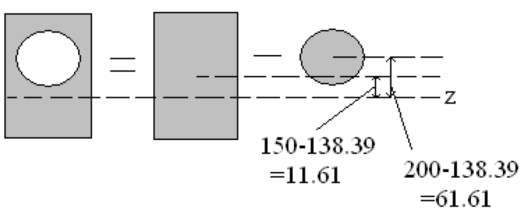
Between BC: $(\sigma_1)_{max} = \dfrac{(3.6x10^3)(.0185)}{2.47x10^{-6}} = 27.0MPa\ (T)$

$$(\sigma_2)_{max} = \frac{(3.6x10^3)(.06152)}{2.47x10^{-6}} = 89.7MPa\ (C)$$

$\Rightarrow (\sigma_c)_{max} =$ ***89.7 Mpa*** $(\sigma_t)_{max} =$ ***50.4 MPa***

e.g. 2
Given: Beam cross section.
Find: c_1, c_2, I





choose z' at bottom ($\bar{y} = c_2$):

$$\bar{y} = \frac{\sum A_i d_i}{\sum A_i} =$$

$$\frac{(200)(100)(50) + [(200)(200) - \pi/4(120)^2](200)}{(300)(200) - \pi/4(120)^2}$$

$= 138.39mm$

where (200)(100) is the area of the lower third and $(200)(200) - \pi/4(120)^2$ *is the area of the upper two-thirds.*

note: The upper two-thirds has height of 200mm. Looking at the dimensions on the figure, we can see that there is a height of 40mm above and 40mm below the cut-out circle within this upper two-thirds block. This symmetry explains why the last term in the numerator, d_i*, is 200 (goes from z' to the midpoint of the circle).*

OR

$$\bar{y} = \frac{(300)(200)(150) - \pi/4(120)^2(200)}{(300)(200) - \pi/4(120)^2} = 138.39mm$$

where (300)(200) is the area of the solid rectangle and $\pi/4(120)^2$ *is the area of the circle.*

$$(I_{rect})_z = \frac{1}{12}(200)(300)^3 + (300)(200)(11.61)^2$$

$$(I_{circ})_z = \frac{\pi}{64}(120)^4 + \pi/4(120)^2(61.61)^2$$

$$I_z = (I_{rect})_z - (I_{circ})_z = 4.908x10^8\ mm^4$$

<u>Bending stress design examples</u>

Bending stress is not related to area and depends on more than one dimension (usually). It turns out, not surprisingly, that for a rectangular section in bending, $A \to 0$ as $h \to \infty$

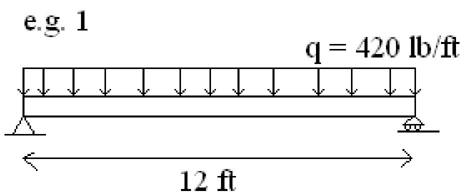
and b $\rightarrow 0$. This is not surprising since we know that sections such as the wide-flange section, with the majority of material away from the neutral axis, are most cost effective. Design is best done using tables of common sections.

Often $\frac{I}{c_1}$ and $\frac{I}{c_2}$ are written as S_1 and S_2, where S = "section modulus." This simplifies the use of tables.

e.g. 1

Given: Wood beam with rectangular cross-sect, is subjected to the load shown. density $= 35\ \frac{lb}{ft^3}$ $\quad \sigma_{allow} = 1800\,psi$.

Find: Suitable size beam (base x height) from appendix A.

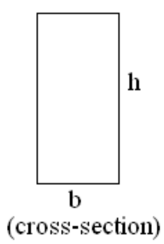


Simply supported beam (pin + roller), uniform load.

$$\Rightarrow M_{max} = \frac{qL^2}{8} = \frac{(420\,{}^{lb}\!/_{ft})(12\,ft)^2(12{}^{in}\!/_{ft})}{8}$$

*=90,720 lb*in (located at midpoint, skipped work)*

$$\sigma_{allow} = \frac{M_{max}}{S} \Rightarrow S = \frac{90{,}720}{1800} = 50.40in^3$$



From Appendix A, choose the lightest (smallest cross-sect area) beam that has a section modulus S of at least 50.40 $\rightarrow$ *choose 3"x12". (3"x12" nominal dimensions, 2.5"x11.25" actual dimensions, s=52.73* in^3 *) But we still have to include the beam's own weight:*

Beam weight = (area)(density)= $\left(\frac{2.5in}{12\,{}^{in}\!/_{ft}} \, x \, \frac{11.25in}{12\,{}^{in}\!/_{ft}}\right)(35\,{}^{lb}\!/_{ft^3}) = 6.8\,{}^{lb}\!/_{ft}$

$$S = \frac{90{,}720 + \frac{(6.8)(12)^2(12)}{8}}{1800} = 51.22in^3 \text{ or } S = (50.40)\left(\frac{426.8}{420}\right) = 51.22in^3$$

This is still smaller than the section modulus for the ***3 x 12 in.*** *beam, so that size is satisfactory.*

note: If $c_1 \neq c_2$, *then the problem is more complicated, but still follows the same basic process.*

note: We have ignored the phenomenon known as "lateral torsional buckling", which will be emphasized in the outline on steel design later on.

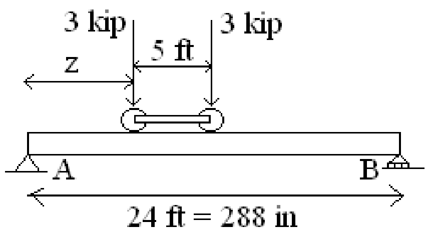
e.g. 2

Given: beam supports the two-wheeled vehicle shown.

It may occupy any position on the beam. $\sigma_{allow} = 21.4ksi$

Find: M_{max} *and the corresponding* S_{min}.

e.g. 2



In terms of arbitrary distance z from the left:
Support reactions:

$$+\circlearrowleft \sum M_A : B(288) - 3(z) - 3(z+60) = 0$$

$$\Rightarrow B = \frac{6z+180}{288} kip$$

$$+\uparrow \sum F_y : A + \frac{6z+180}{288} - 6 = 0 \Rightarrow A = \frac{1548-6z}{288} kip$$

$$x = 0^- : V(0^-) = M(0^-) = 0$$

$$x = 0^+ : V(0^+) = \frac{1548-6z}{288} \quad M(0^+) = 0$$

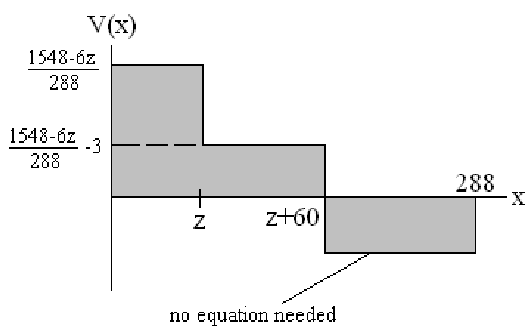
$$0^+ \le x \le z^- : V(x) = \frac{1548-6z}{288} \quad M(x) = \int_0^x \frac{1548-6z}{288} dx = \frac{1548-6z}{288} x$$

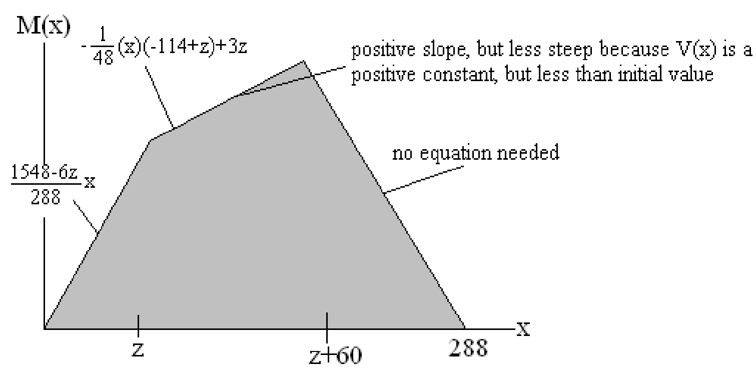
$$x = z^- : V(z^-) = \frac{1548-6z}{288} \quad M(z^-) = \frac{1548-6z}{288} z$$

$$x = z^+ : V(z^+) = \frac{1548-6z}{288} - 3 \quad M(z^+) = \frac{1548-6z}{288} z$$

$$z^+ \le x \le (z+60)^- : V(x) = \frac{1548-6z}{288} - 3 \quad M(x) = \frac{1548-6z}{288} z + \int_z^x \frac{1548-6z}{288} - 3dx$$

$$= -\frac{1}{48}(x)(-114+z) + 3z$$





$$M_{max} = M(z+60) = -\frac{1}{48}(z+60)(-114+z) + 3z \qquad z \; \varepsilon \; [0, 288-60]$$

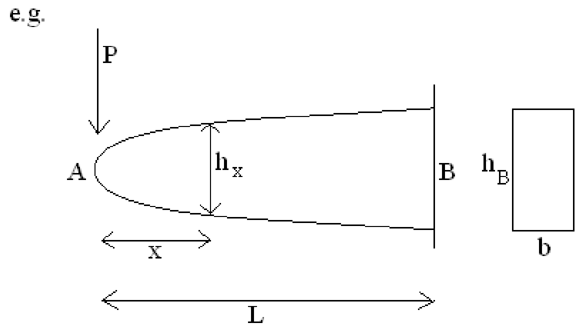
$$\Rightarrow \boldsymbol{z_{max} = 99in,\ M_{max} = 346.7 kip * in \quad S_{min} = \frac{346.7}{21.4} = 16.2 in^3}$$

note: could then use a table to choose an efficient beam size.

Tapered beams

To really minimize the amount of material, the cross-sect dimensions can be varied so as to develop the maximum allowable bending stress at every section.

e.g.
Given: cantilevered beam with point load shown.
Find: h_x *so that* $\sigma = \sigma_{allow}$ *at every cross-section.*

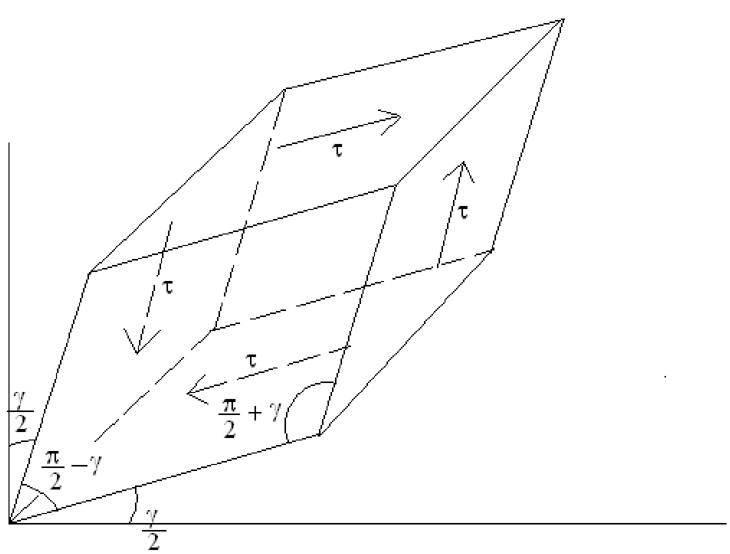


$$\sigma_{allow} = \frac{M_x\left(\frac{h_x}{2}\right)}{\left(\frac{1}{12}bh_x^3\right)} = \frac{6Px}{bh_x^2} \Rightarrow \boldsymbol{h_x = \sqrt{\frac{6Px}{b\sigma_{allow}}}}$$

note: If $c_1 \neq c_2$*, then the problem gets a bit more complicated.*
note: angle of taper must not be too large.

Shear

Shear deformation



If a shear force τ acts on the upper face, each side must have an equal shear force (in the directions shown) for equilibrium.

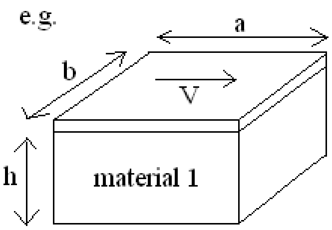
The shear forces create a distortion as shown. γ is called the shear strain (radians).

Shear stress – strain diagrams appear similar to the axial diagram that was shown at the beginning of this chapter.

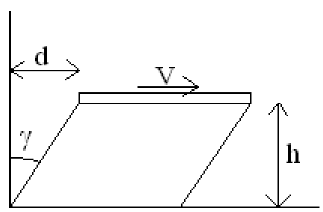
$\tau = G\gamma$
where G = Shear Modulus of Elasticity (material property)

$\gamma = \frac{V}{AG}$, where A = shear area note: $G = \dfrac{E}{2(1+\nu)}$ (skipped proof)

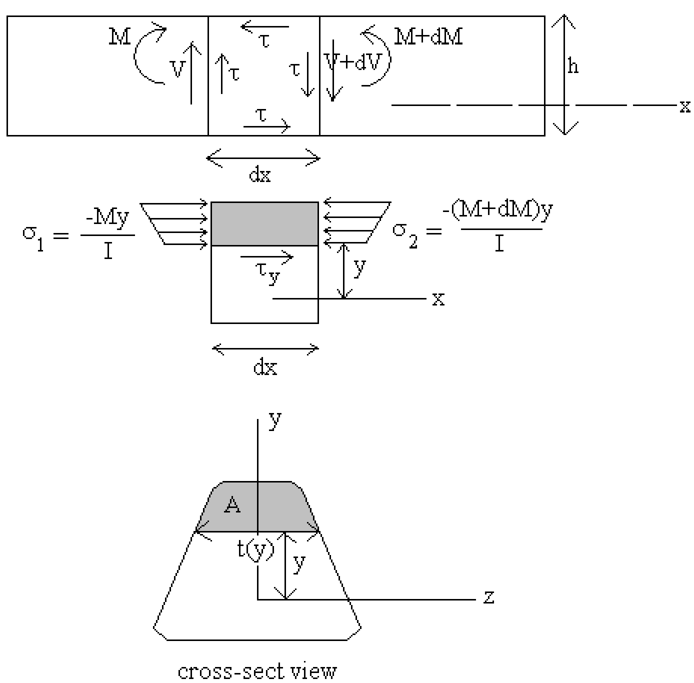
e.g.
Given: "bearing pad" with dimensions shown, subjected to force shown.
Find: τ, γ, and d.



$$\boldsymbol{\tau = \frac{V}{ab} \quad \gamma = \frac{V}{abG_1} \quad d = h\tan\gamma = h\tan\left(\frac{V}{abG_1}\right)}$$



<u>Shear stress in flexure</u>



From equilibrium of shear, the shear stress in the vertical direction is matched with an equal shear stress in the horizontal direction. And, from equilibrium of force in the x direction,

$$(\tau_y)*[t(y)dx] = \int_A \left[\frac{(M+dM)y}{I}\right](dA) - \int_A \left[\frac{My}{I}\right](dA)$$

The units match, since we have:

$$(F/A)*[A] = [F/A](A) - [F/A](A)$$

The area A is the area shaded, <u>not</u> the entire cross-sectional area.

$$\tau_y = \frac{dM}{dx}\frac{1}{I*t(y)}\int_A y\,dA$$

$$\boldsymbol{\tau_y = \frac{VQ_y}{I*t(y)}} \text{ (General Formula)} \quad \text{where } Q = \text{"first moment"} = \int_A y\,dA$$

note: In pure bending, V = 0, so $\tau = 0$. Also, M + dM = M in that case, so $\tau = 0$.

For rectangular cross-sect, t(y) = b (base) and $Q_y = \frac{b}{2}(\frac{h^2}{4} - y^2)$

$\tau_y = \frac{3V(h^2 - 4y^2)}{2bh^3}$ (rect cross-sect) (skipped work)

τ_{max} occurs at y=0, which is the neutral axis.

$\boldsymbol{\tau_{max} = \frac{3V}{2A}}$ (rect cross-sect) A=bh

note: Although τ was calculated as being horizontal, there *must be* vertical shear that is equal, so V_{max} is determined from the SFD. Area A is always the cross-sectional area.

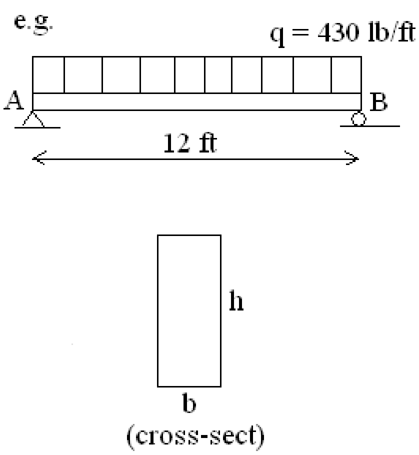
note: Now it is possible to optimize the bending stress for a rect sect, although designers still usually use tables.

e.g.

Given: Wood beam with rectangular cross-sect, is subjected to load shown.

$\tau_{allow} = 200\,psi, \sigma_{allow} = 1800\,psi$

Find: Optimal beam size (assume beam weight already included in load q).



$$M_{max} = \frac{qL^2}{8} = 92{,}880\ lb*in$$

$$Supports:\quad A = B = \frac{430(12)}{2} = 2580lb = V_{max}$$

$$1800 = \sigma_{allow} = \frac{My}{I} = \frac{92880(h/2)}{(\frac{1}{12}bh^3)} = \frac{557280}{bh^2} \qquad (1)$$

$$200 = \tau_{allow} = \frac{3V}{2A} = \frac{3(2580)}{2(bh)} = \frac{3870}{bh} \qquad (2)$$

2 eq, 2 unknowns;

h=16", b=1.21" $(A_{min} = bh = 19.35in^2)$

note: h>>b, as expected.

note: (compare to e.g. 1 of the "Bending stress design examples" section) Even though an overly large allowance for the beam's own weight was provided, and very small τ_{allow}*, this beam was still about 2/3 the weight of the beam chosen in e.g. 1. Of course, this is also largely due to the limited selection of available beams in the Appendix A.*

For circular cross-sect, τ is complicated away from the neutral axis. But, we can still find τ_{max} which has been proven experimentally to be located at the neutral axis:

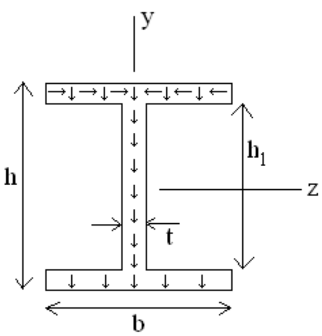
t(0) = d (diameter) and $Q_0 = \frac{1}{12}d^3$

$\tau_{max} = \frac{4V}{3A}$ (solid shaft) (skipped work) $A = \pi r^2$

$\tau_{max} = \frac{4V}{3A}\left(\frac{r_2^{\ 2} + r_2 r_1 + r_1^{\ 2}}{r_2^{\ 2} + r_1^{\ 2}}\right)$ (hollow tube) $A = \pi(r_2^{\ 2} - r_1^{\ 2})$

note: Just like a rect sect, it is now possible to optimize a tubular section, although the use of a table is more practical. Just make sure $\tau \le \tau_{max}$ after a size with appropriate section modulus has been chosen from the table.

Wide-flange cross section:



cross-sect view

Although the *resultant* forces are located in the xy plane, there are forces distributed all over the upper flange. This creates a bending moment in the flange about the *x* axis and accompanying bending stresses and horizontal shear stresses. The web has only vertical shear stresses which can easily be determined. For the web, t(y) = t (web thickness) and $Q_y = \frac{b}{8}(h^2 - h_1^{\ 2}) + \frac{t}{8}(h_1^{\ 2} - 4y^2)$,

$I = \frac{1}{12}(bh^3 - bh_1^{\ 3} + th_1^{\ 3})$.

- see next example for Q calculation of odd shape.

$\tau_y = \frac{3V[b(h^2 - h_1^{\ 2}) + t(h_1^{\ 2} - 4y^2)]}{2t(bh^3 - bh_1^{\ 3} + th_1^{\ 3})}$ (wide-flange beam)

(skipped work)

τ_{max} occurs at the neutral axis

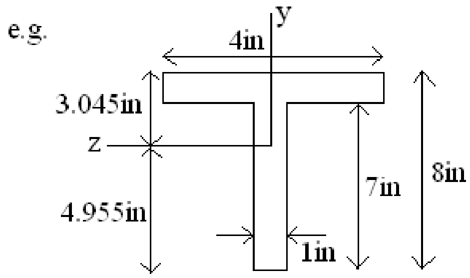
$\tau_{max} = \frac{3V(bh^2 - bh_1^{\ 2} + th_1^{\ 2})}{2t(bh^3 - bh_1^{\ 3} + th_1^{\ 3})}$

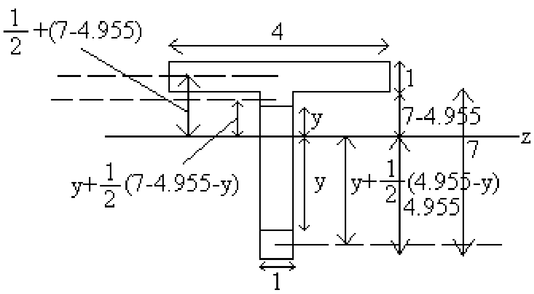
note: a typical wide-flange beam design would be to design for σ_{allow} from a table, and then check $\tau \le \tau_{allow}$.

$\tau_{ave} = \frac{V}{th_1}$ and in this case is close to τ_{max} (within 10% plus or minus), so τ_{ave} is sometimes used in practice. We will learn methods for calculating shear, which are more often used in practice, in later chapters on concrete design and steel design.

note: τ_{ave} was also used in the design of bolted connections in chapter 1.

e.g.
Given: Location of neutral axis. $I = 69.65in^4, V_{max} = 10{,}000lb$.
Find: τ_{max} *in web.*





cross-sect views

First moment from $Q_y = \sum A_i d_i$

$A_i =$ *area a distance* $\geq y$ *away from neutral axis.*

$d_i =$ *distance from* A_i *(*A_i *neutral axis) to z.*

Choose y in web <u>above</u> neutral axis:

$$Q_y = (1x(7-4.955-y))[y+\frac{1}{2}(7-4.955-y)] +$$

$$(1x4)[\frac{1}{2}+(7-4.955)] = 12.3-\frac{1}{2}y^2$$

where $1x(7-4.955-y)$ *is the shaded area at the top of the web, and (1x4) is the flange area.*

OR

Choose y in web <u>below</u> neutral axis:

$$Q_y = [1x(4.955-y)][y+\frac{1}{2}(4.955-y)] =$$

$$12.3-\frac{1}{2}y^2$$

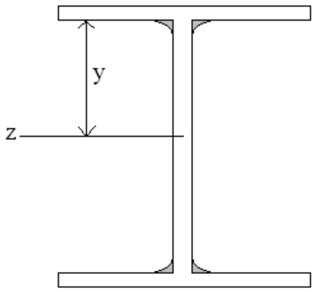
as expected, where $[1x(4.955-y)]$ *is the shaded area at the bottom of the web.* Q_{max} *occurs when y = 0. Since t(y) is constant,* τ_{max} *also occurs when y = 0 (a.k.a. the neutral axis z)*

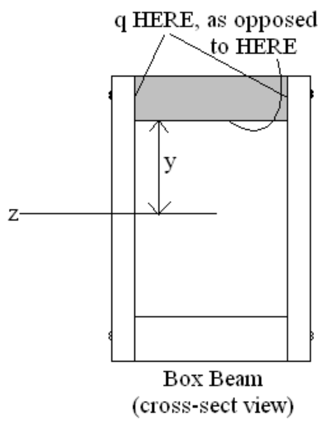
$$\tau_{max} = \frac{VQ}{It} = \frac{10000(12.3)}{69.65(1)} = 1.8ksi$$

note: τ_{max} *occurs at the neutral axis for almost <u>any</u> cross-section.*

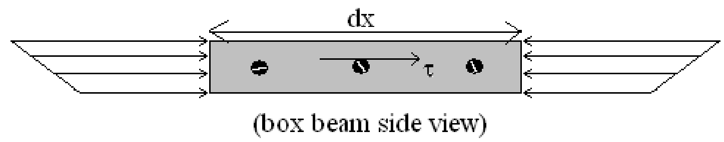
Shear flow

Shear flow = $q_y = \tau * t(y)$ or $\mathbf{q_y = \frac{VQ_y}{I}}$ $(\frac{\text{force}}{\text{dist}})$ where y is the distance at which there is to be nailing or welding. Q would typically be found after a beam size has been chosen. The strength of a weld is usually specified in terms of force per unit distance, as we will see later on in the outline on steel design. So, the required weld strength = q (or $\frac{q}{2}$ for the picture shown below). Nail and screw strength is usually specified in units of force F. Nail spacing = s = $\frac{F}{q}$, where F (allowable force of the screw or nail) can be looked up for a given nail type.



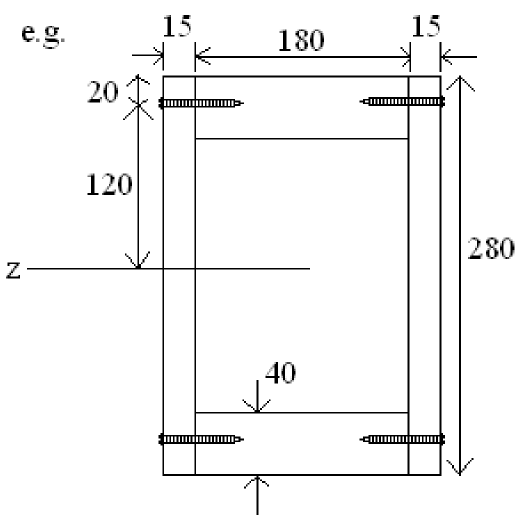


For a “Box beam”, you can think of the segment



Since there is negligible σ about the x axis for a box beam, we need only consider $\sum F_x = 0$, which shows that τ shown differs from the τ we’ve been calculating only in that t(y) is the vertical cross-sect thickness (same Q and I (about the z axis) can be used). Since shear *flow* does not depend on thickness, there is no difference in q. Q would be integrated over the shaded region above.

e.g.
Given: Box beam shown subjected to shear force V = 10.5 kN. Allowable screw force (shear force for screw) F = 800 N.
Find: Screw spacing s.



$$Q = (180)(40)(120) = 864x10^3\, mm^3$$

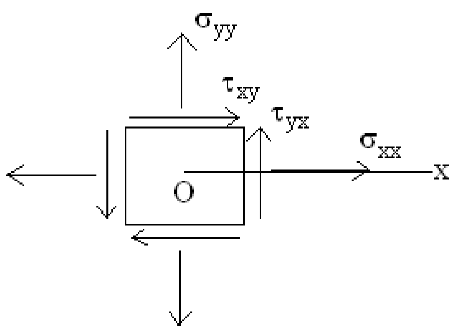
$$I = \frac{1}{12}(180 + 2*15)(280)^3 - \frac{1}{12}(180)(280 - 2*40)^3$$

$$= 264.2x10^6\, mm^4$$

$$q = \frac{VQ}{I} = \frac{(10.5x10^3)(864x10^3)}{(264.2x10^6)} = 34.3\, N/mm$$

$$S = \frac{2F}{q} = \frac{2(800)}{34.3} = \mathbf{46.6mm}$$

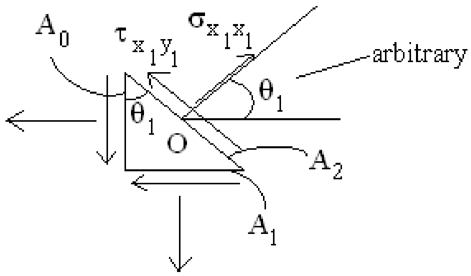
Principal stresses



Stresses are positive if <u>positive</u> face – <u>positive</u> direction or <u>negative</u> face – <u>negative</u> direction.
(All stresses shown are positive with respect to these axes)

$$\cos\theta_1 = \frac{A_0}{A_2} \Rightarrow A_2 = A_0 \sec\theta_1$$

$$\tan\theta_1 = \frac{A_1}{A_0} \Rightarrow A_1 = A_0 \tan\theta_1$$



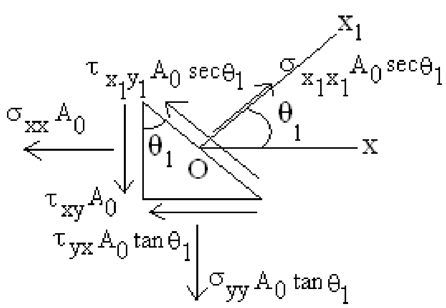
stresses

From $\sum F_x = 0$ and $\sum F_y = 0$,

$$\boldsymbol{\sigma_{x1x1} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta_1 + \tau_{xy}\sin 2\theta_1}$$

$$\boldsymbol{\tau_{x1y1} = \frac{-(\sigma_{xx} - \sigma_{yy})}{2}\sin 2\theta_1 + \tau_{xy}\cos 2\theta_1}$$

$\theta_1 \;\varepsilon\; [0°, 180°]$

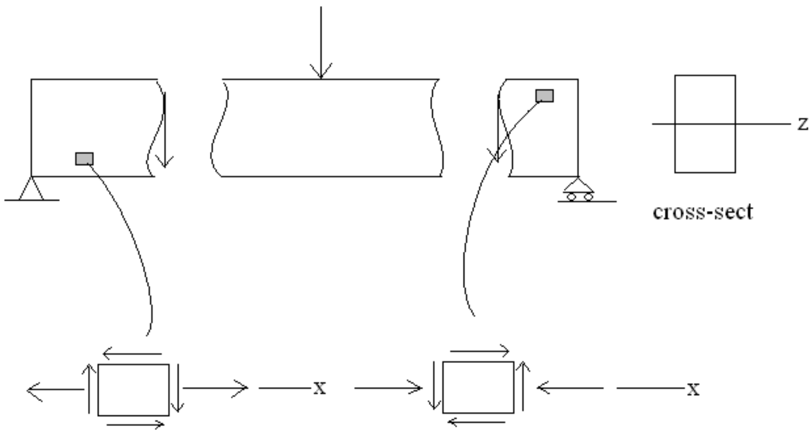


forces

These are the “transformation equations” for plane stress. σ_{max} and τ_{max} from these equations are the <u>true</u> maximum stresses in a beam (except for the special case where they occur out-of-plane). σ_{max} and τ_{max} may occur at a location of $(\sigma_{xx})_{max}, (\sigma_{yy})_{max}, (\tau_{xy})_{max}$, or may occur at a location where none of the above are maximized.

note: σ_{yy} for a given region in a beam is the distributed load q divided by the cross-sect thickness t at that location.

note: σ_{yy} usually compressive (negative in the above equations) since our distributed loads act downward. σ_{xx} direction determined from bending stress and external axial load $(\frac{P}{A} + \frac{My}{I})$. τ direction determined from inspection of the internal vertical equilibrium (NOT SFD) (see below).



τ reversed because location is to the right of the point load.

σ reversed because section chosen is in upper portion of beam.

note: The beams of chapter five usually contain all three forces. Since σ_{yy} depends on x (distance along beam) and y (in relation to neutral axis), σ_{xx} depends on x and y, τ_{xy} depends on x and y, and θ_1 also varies between 0 and 180°, finding the exact location and angle of σ_{max} and τ_{max} can usually only be approached through trial and error using the transformation equations. For design, transformation equations are accounted for in the safety factor, but can be checked as follows.

Principal Angles

The following is useful assuming that a location O within a beam has been chosen and $\tau_{xy}, \sigma_{xx}, \sigma_{yy}$ are known.

From $\frac{d\sigma_{x1x1}}{d\theta_1} = 0$, $\mathbf{\tan 2(\theta_p) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}}$ (critical angles for normal – "principal stress")

Two solutions: $\theta_p\ \varepsilon\,[0, 90°]$ and $\theta_p\ \varepsilon\,[90, 180°]$ which correspond to θ_{p1} and θ_{p2} though not necessarily in that order. θ_{p1} and θ_{p2} differ by 90°.

$$\boldsymbol{\sigma_{xp1xp1} = (\sigma_{x1x1})_{max} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^{\,2}}} \quad \text{(skipped work)}$$

$$\boldsymbol{\sigma_{xp2xp2} = (\sigma_{x1x1})_{min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^{\,2}}}$$

(could be greater magnitude than $(\sigma_{x1x1})_{"max"}$)

note: $\tau_{xp1xp1} = \tau_{xp2xp2} = 0$ (proof Mohr's Circle – see next section)

note: The true min and max normal stress could be located "out-of-plane" (not calculated)

From $\frac{d\tau_{x1y1}}{d\theta_1} = 0$, $\mathbf{\tan 2(\theta_s) = \frac{-(\sigma_{xx} - \sigma_{yy})}{2\tau_{xy}}}$ (critical angles for shear stress)

Two solutions: $\theta_s \;\varepsilon\, [0, 90°]$ and $\theta_s \;\varepsilon\, [90, 180°]$ which correspond to θ_{s1} and θ_{s2} though not necessarily in that order. θ_{s1} and θ_{s2} differ by 90°.

$$\boldsymbol{\tau_{xs1ys1} = (\tau_{x1y1})_{max} = \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^{\;2}}} \quad \textbf{OR} \quad \boldsymbol{\frac{\sigma_{xp1xp1} - \sigma_{xp2xp2}}{2}}$$

$$\tau_{xs2ys2} = (\tau_{x1y1})_{min} = -(\tau_{x1y1})_{max}$$

note: $\sigma_{ave} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \sigma_{xs1xs1} = \sigma_{xs2xs2}$ (Proof Mohr's Circle – see next section)

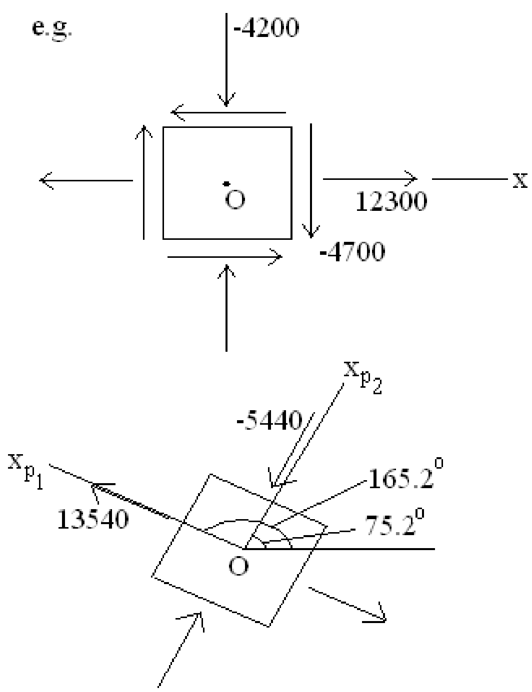
note: $\theta_{s1} = \theta_{p1} - 45°$ (Proof Mohr's Circle)

note: The true min and max shear stress is located out-of-plane if σ_{xp1xp1} and σ_{xp2xp2} have the same sign: $[(\tau_{max/min})_{about\ xp1} = \pm\frac{\sigma_{xp2xp2}}{2}$ and $(\tau_{max/min})_{about\ xp2} = \pm\frac{\sigma_{xp1xp1}}{2}]$.

e.g.

Given: $\sigma_{xx} = 12300\, psi \quad \sigma_{yy} = -4200\, psi \quad \tau_{xy} = -4700\, psi$

Find: $\sigma_{xp1xp1}, \sigma_{xp2xp2}, \tau_{xs1ys1}, \tau_{xs2ys2}$



$$tan\, 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \Rightarrow 2\theta_p = -29.67° = 330.33°$$

$\theta_p = 165.2°$ *and* $75.2°$

$$\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} cos\, 2(165.2) + \tau_{xy}\, sin\, 2(165.2)$$

= ***13540 psi*** $= \sigma_{xp1xp1}$

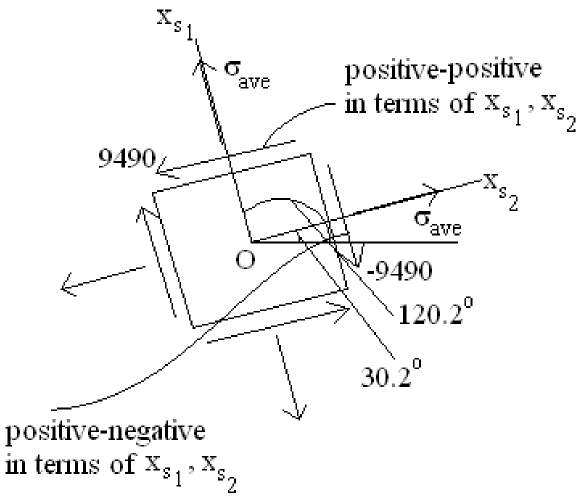
$$\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} cos\, 2(75.2) + \tau_{xy}\, sin\, 2(75.2)$$

= ***-5440 psi*** $= \sigma_{xp2xp2}$

check:

$$\frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^{\;2}}$$

= *-5440 and 13540, as expected*



$$\tan 2\theta_s = \frac{-(\sigma_{xx} - \sigma_{yy})}{2\tau_{xy}} \Rightarrow 2\theta_s = 60.4°$$

$\theta_s = 30.2°$ and $120.2°$

$$\frac{-(\sigma_{xx} - \sigma_{yy})}{2} \sin 2(30.2) + \tau_{xy} \cos 2(30.2)$$

$= \mathbf{-9490\ psi} = \tau_{xs2ys2}$

$$\frac{-(\sigma_{xx} - \sigma_{yy})}{2} \sin 2(120.2) + \tau_{xy} \cos 2(120.2)$$

$= \mathbf{9490\ psi} = \tau_{xs1ys1}$

check: $\pm\sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^{\ 2}} = \pm 9490$

note: $\theta_{s1} = \theta_{p1} - 45° = 165.2° - 45° = 120.2°$

note: $\tau_{xs1ys1} = -\tau_{xs2ys2} = \frac{\sigma_{xp1xp1} - \sigma_{xp2xp2}}{2} = \frac{13540 - (-5440)}{2} = 9490$

note: $\tau_{xp1yp1} = \tau_{xp2yp2} = 0$ and $\sigma_{xs1xs1} = \sigma_{xs2xs2} = \sigma_{ave}$ could also be shown easily

Mohr's Circle

$$\sigma_{x1x1} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta_1 + \tau_{xy} \sin 2\theta_1$$

and

$$\tau_{x1y1} = \frac{-(\sigma_{xx} - \sigma_{yy})}{2} \sin 2\theta_1 + \tau_{xy} \cos 2\theta_1$$

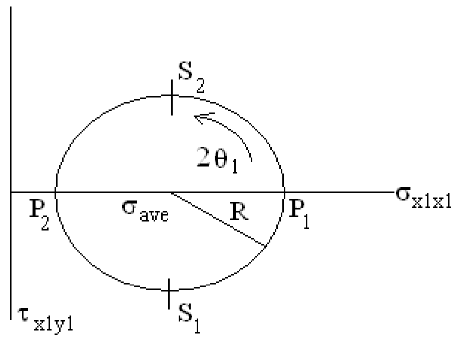
are the parametric equations of a circle.

Manipulation: Bring $\frac{\sigma_{xx} + \sigma_{yy}}{2}$ to the left side of the top equation, square both sides of the equation, and then add the two equations;

$(\sigma_{x1x1} - \sigma_{ave})^2 + \tau_{x1y1}^{\ 2} = R^2 \Rightarrow R = \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^{\ 2}}$ is the algebraic equation of a circle.

Knowing $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$:

- we can now find τ_{x1y1} directly from σ_{x1x1} without knowing θ (and vice-versa).
- we can construct the circle with an accurate scale and immediately see all values of τ_{x1y1} and σ_{x1x1} and their corresponding θ_1 (by measuring $2\theta_1$ with a protractor).



note: τ_{x1y1} is positive downward, and $\theta = 0°$ does <u>NOT</u> necessarily start at the σ_{x1x1} axis.

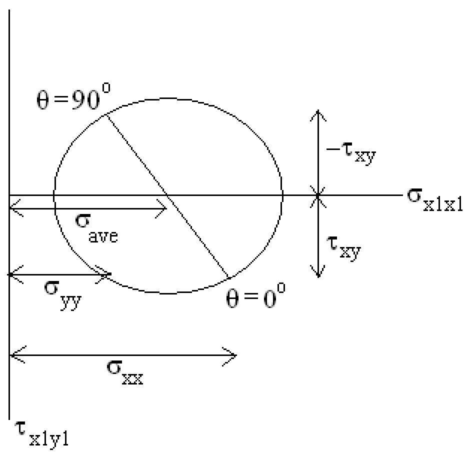
note: @ P_1 and P_2, $\tau = 0$

@ S_1 and S_2, $\sigma = \sigma_{ave}$

$\theta_{s1} = \theta_{p1} - 45°$, all as expected.

Also,

$$(\tau_{x1y1})_{max} = \text{radius} = \frac{\text{diameter}}{2} = \frac{(\sigma_{x1x1})_{max} - (\sigma_{x1x1})_{min}}{2}$$



Procedure for drawing Mohr's Circle;

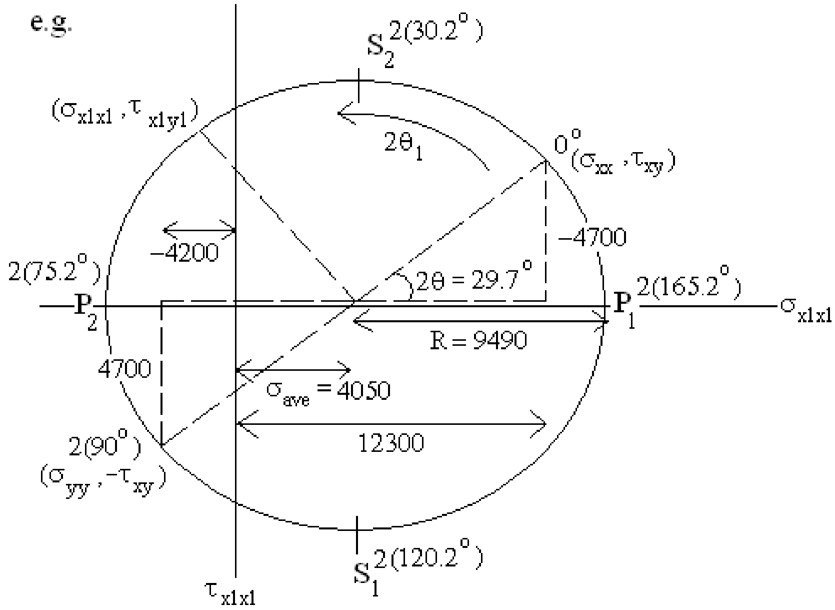
First, draw axes.
Second, the center of the circle corresponds to $(\sigma_{ave}, 0)$.

- A straight line through the center connecting (σ_{xx}, τ_{xy}) and $(\sigma_{yy}, -\tau_{xy})$ is the circle's diameter. (2R also = diameter)
- $\theta = 0$ corresponds to (σ_{xx}, τ_{xy}), $2\theta = 180$ corresponds to $(\sigma_{yy}, -\tau_{xy})$
- Draw the circle
- Find τ and σ values of interest directly from circle (if drawn with accurate scale), or from transformation equations, or using trig.

If we know the stresses σ_{x1x1}, σ_{y1y1}, and τ_{x1y1}, at a known angle θ_1, we can construct the circle first in terms of these stresses and then move <u>clockwise</u> $2\theta_1$ for σ_{xx}, σ_{yy}, and τ_{xy}.

e.g.
Given: $\sigma_{xx} = 12300\,psi$ $\sigma_{yy} = -4200\,psi$ $\tau_{xy} = -4700\,psi$
Find: $\theta_{p1}, \theta_{p2}, \theta_{s1}, \theta_{s2}, \sigma_{xp1xp1}, \sigma_{xp2xp2}, \tau_{xs1ys1}, \tau_{xs2ys2}$ *AND stresses at* $\theta_1 = 45°$

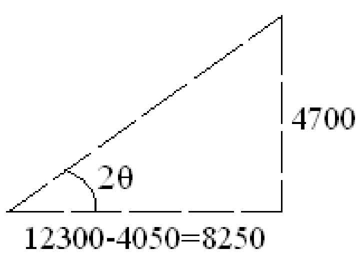
e.g.



(not to scale)

$$\sigma_{ave} = \frac{12300 + (-4200)}{2} = 4050$$

$$R = \sqrt{\left(\frac{12300 - (-4200)}{2}\right)^2 + (-4700)^2} = 9490$$



$$2\theta = \tan^{-1}\left(\frac{4700}{8250}\right) = 29.7^\circ$$

$$2\theta_{p2} = 180 - 29.7 = 150.3^\circ$$

$$\theta_{p2} = 75.2^\circ$$

$\sigma_{xp2xp2} = \sigma_{ave} - R =$ **-5440 psi**

$2\theta_{p1} = 360 - 29.7 = 330.3^\circ \quad \theta_{p1} = 165.2^\circ$

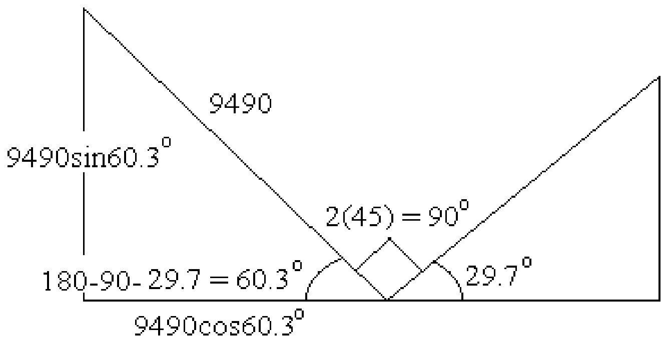
$\sigma_{xp1xp1} = \sigma_{ave} + R =$ **13540 psi**

$2\theta_{s2} = 90 - 29.7 = 60.3^\circ \quad \theta_{s2} = 30.2^\circ$

$\tau_{xs2ys2} = -R =$ **-9490 psi**

$2\theta_{s1} = 270 - 29.7 = 240.3^\circ \quad \theta_{s1} = 120.2^\circ$

$\tau_{xs1ys1} = R =$ **9490 psi**

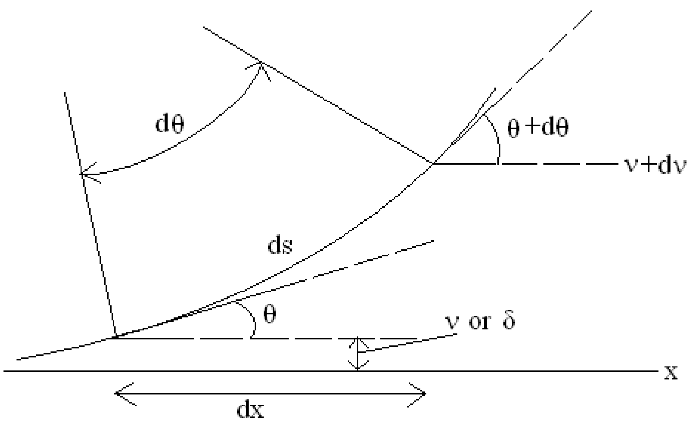


$\tau_{x1y1} = -9490 \sin 60.3° = \mathbf{-8245\ psi}$

$\sigma_{x1x1} = 4045 - 9490 \cos 60.3° = \mathbf{-660\ psi}$

- compare solutions with the previous example

Beam deflections and rotations



$$\kappa = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \text{ and } \frac{d\upsilon}{dx} = \tan\theta \approx \theta$$

$$\text{So, } \frac{d\theta}{dx} = \frac{d^2\upsilon}{dx^2} = \kappa = \frac{M}{EI}$$

$$\Rightarrow EI\frac{d^2\upsilon}{dx^2} = M$$

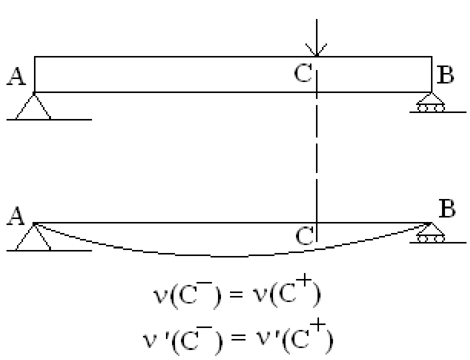
Substitute BMD function(s) for M, and integrate.

$$\mathbf{EI\upsilon'''' = -q \qquad EI\upsilon''' = V \qquad EI\upsilon'' = M \qquad EI\upsilon' = EI\theta \qquad EI\upsilon = EI\delta}$$

q, V, M should be non-zero constant, or functions of x
(If non-prismatic beam, then I also depends on x)

note: small deflections only

Solving the second-order bending moment equation $EI\upsilon'' = M$, yields two constants of integration (for each segment of a beam). We need two sets of initial conditions (for each segment). There are always enough to choose from, if the system is statically determinate:

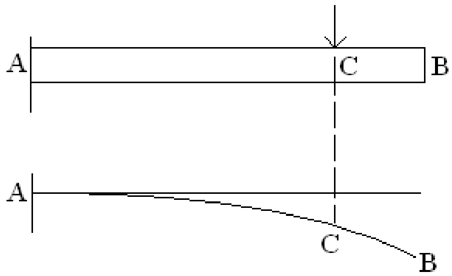


Simply-supported beam shown:

- boundary conditions:
  - $\upsilon(A) = 0$
  - $\upsilon(B) = 0$

- continuity conditions:
  - $\upsilon(C^-) = \upsilon(C^+)$
  - $\upsilon'(C^-) = \upsilon'(C^+)$
- symmetry conditions:
  none

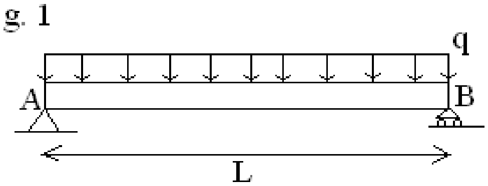
Cantilevered beam shown:



- boundary conditions:
  - $\upsilon(A) = 0$
  - $\upsilon'(A) = 0$
- continuity conditions:
  - $\upsilon(C^-) = \upsilon(C^+)$
  - $\upsilon'(C^-) = \upsilon'(C^+)$
- symmetry conditions:
  none

e.g. 1
Find: Deflection curve $\upsilon, \delta_{max}, \theta_{max}$.

e.g. 1



$$M = \frac{qL}{2}(x) - qx(\frac{x}{2}) = \frac{qLx}{2} - \frac{qx^2}{2} \text{ (skipped work)}$$

$$EI\upsilon'' = \frac{qLx}{2} - \frac{qx^2}{2}$$

$$EI\int \upsilon''\,dx = EI\upsilon' = \int \frac{qLx}{2}dx - \int \frac{qx^2}{2}dx$$

$$= \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1$$

$$EI\int \upsilon'\,dx = EI\upsilon = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1 x + C_2$$

symmetry condition: $\upsilon'(\frac{L}{2}) = 0$

$$0 = \frac{qL}{4}(\frac{L}{2})^2 - \frac{q}{6}(\frac{L}{2})^3 + C_1 \Rightarrow C_1 = -\frac{qL^3}{24}$$

$$EI\upsilon = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3 x}{24} + C_2$$

boundary condition: $\upsilon(L) = 0 \text{ or } \upsilon(0) = 0$

$C_2 = 0$

$$\boldsymbol{\upsilon = -\frac{qx}{24EI}(L^3 - 2Lx^2 + x^3)} \quad \delta_{max} \text{ located at } \upsilon' = 0$$

$$\frac{qLx^2}{4}-\frac{qx^3}{6}-\frac{qL^3}{24}=0 \Rightarrow x=\frac{L}{2} \text{ as expected}$$

$$\delta_{max}=\upsilon(\frac{L}{2})=\frac{\mathbf{5qL^4}}{\mathbf{384EI}} \quad \theta_{max} \text{ located at } \upsilon''=0$$

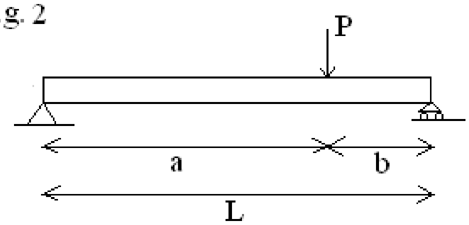
$$\frac{qLx}{2}-\frac{qx^2}{2}=0 \Rightarrow x=0 \text{ or } L \text{ as expected}$$

$$\theta_{max}=\upsilon'(L)=\left|\upsilon'(0)\right|=\frac{\mathbf{qL^3}}{\mathbf{24EI}}$$

e.g. 2

Find: $\upsilon_1 \ \varepsilon \ 0 \le x \le a^-$, $\upsilon_2 \ \varepsilon \ a^+ \le x \le L$, θ_1 , θ_2 , δ_{max}

e.g. 2



$$0 \le x \le a^- : M=\frac{Pbx}{L} \quad \text{(skipped work)}$$

$$a^+ \le x \le L : M=\frac{Pbx}{L}-P(x-a) \quad \text{(skipped work)}$$

$$EI\upsilon_1''=\frac{Pbx}{L} \qquad EI\upsilon_2''=\frac{Pbx}{L}-P(x-a)$$

$$EI\upsilon_1'=\frac{Pbx^2}{2L}+C_1 \qquad EI\upsilon_2'=\frac{Pbx^2}{2L}-\frac{P(x-a)^2}{2}+C_2$$

$$EI\upsilon_1=\frac{Pbx^3}{6L}+C_1x+C_3 \qquad EI\upsilon_2=\frac{Pbx^3}{6L}-\frac{P(x-a)^3}{6}+C_2x+C_4$$

continuity condition: $\upsilon_1'(a^-)=\upsilon_2'(a^+)$

$$\frac{Pba^2}{2L}+C_1=\frac{Pba^2}{2L}-\frac{P(a-a)^2}{2}+C_2 \quad \text{cancelling terms} \Rightarrow C_1=C_2$$

continuity condition: $\upsilon_1(a^-)=\upsilon_2(a^+)$

$$\frac{Pba^3}{6L}+C_1a+C_3=\frac{Pba^3}{6L}-\frac{P(a-a)^3}{6}+C_2a+C_4 \quad \text{cancelling terms} \Rightarrow C_3=C_4$$

boundary condition: $\upsilon_1(0)=0$

$$0=\frac{Pb(0)^3}{6L}+C_1(0)+C_3 \Rightarrow C_3=0$$

boundary condition: $\upsilon_2(L)=0$

$$0=\frac{Pb(L)^3}{6L}-\frac{P(L-a)^3}{6}+C_2L+C_4 \quad C_4=0 \Rightarrow C_2=\frac{-Pb(L^2-b^2)}{6L}$$

$0 \le x \le a^- :$ $\qquad$ $a^+ \le x \le L :$

$$\upsilon_1 = \frac{-Pbx}{6LEI}(L^2 - b^2 - x^2) \qquad \upsilon_2 = \frac{-Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI}$$

$$\theta_1 = \upsilon_1' = \frac{-Pb}{6LEI}(L^2 - b^2 - 3x^2) \qquad \theta_2 = \frac{-Pb}{6LEI}(L^2 - b^2 - 3x^2) - \frac{P(x-a)^2}{2EI}$$

For a > b, δ_{max} *obviously* $\varepsilon\,(0, a^-)$.

$$\delta_{max} \text{ at } \upsilon_1' = 0 \Rightarrow x = \sqrt{\frac{L^2 - b^2}{3}} \qquad \delta_{max} = \upsilon_1\left(\sqrt{\frac{L^2 - b^2}{3}}\right) = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,LEI} \quad (a \ge b)$$

note: The special case method for finding $\delta_{midpoint}$ *in the flexure derivation can still be used.*

note: Starting with the bending moment equation always works.

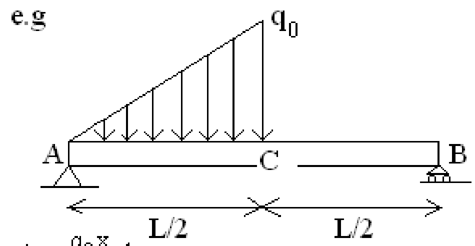
Superposition

For beams with common uniform loads AND point loads, where $\upsilon(x)$ and $\theta(x)$ can be looked up in a table for the cases where each type of loading is acting alone, $\upsilon_{total} = \sum \upsilon$ and $\theta_{total} = \sum \theta$. Values can be found at specific points, or general (in terms of x) formulas can be found.
Superposition can provide a useful shortcut for unusual loads too. But for these loads, it is usually NOT possible to obtain a general formula $\upsilon(x)$ and $\theta(x)$ for the whole beam because point load formulas (which are different for the left side of the load versus the right side) must be summed, and the shortcut involves an infinite number of point loads. (see next example).

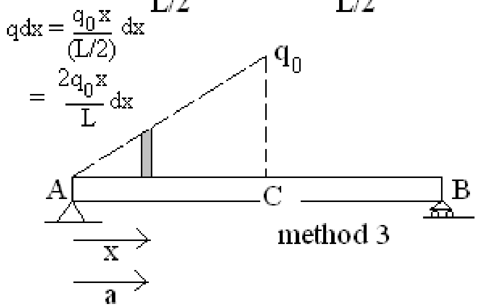
e.g.
Find: δ_c

Method 1: find M(x) and solve $EI\upsilon''$
Method 2: find $-q(x)$ *and solve* $EI\upsilon''''\ \varepsilon\ [A, C]$
Method 3: point load midpoint deflection formula (tabulated in the appendix of many textbooks): $\frac{Pa}{48EI}(3L^2 - 4a^2) \quad b \ge a$ (note: this equation works for all points under the load, i.e. between A and C)

For an arbitrary point under the triangular load, the force P = qdx = $\frac{2q_0 x}{L}dx$ and the distance "a" is "x". The deflection at C is the sum of the deflections caused by each infinitesimal force.



$$\delta_c = \int_0^{L/2} \frac{qx}{48EI}(3L^2 - 4x^2)dx = \boldsymbol{\frac{q_0 L^4}{240EI}}$$



Method 4: point load deflection formula for $a \le x \le L$ *from a previous example:*

"a" is "z" and "b" is "L-z"

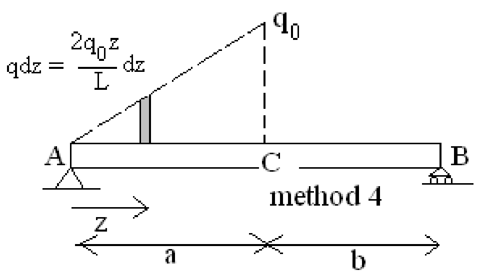
$$\frac{-Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI}$$

$$P = qdx = \frac{2q_0 z}{L}dz$$

$$\upsilon(x)\ \varepsilon\ [C,B] = \int \frac{-q_0 z(L-z)(x)}{3L^2 EI}(L^2 - (L-z)^2 - x^2)$$

$$-\frac{q_0 z(x-z)^3}{3LEI}dz = \frac{q_0 L(3L^3 - 43L^2 x + 60Lx^2 - 20x^3)}{1440EI}$$

$$\upsilon(L/2) = \boldsymbol{\frac{-q_0 L^4}{240EI}}$$

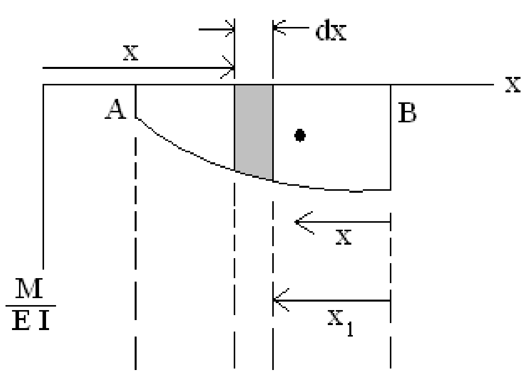


note: If the triangular load starts at a distance "k" away from A, then the lower limit of integration would be k.

note: There is no <u>*easy*</u> *way to obtain a* <u>*general*</u> *formula for the beam, which includes* $\upsilon(x)\ \varepsilon\ [A,C]$*, because under the triangular load, the location of* υ *is to the left of some of the "point loads" and to the right of others (two separate formulas).*

<u>Moment-Area Method</u>

Just like load equations, this method is particularly useful for cantilevered beams.

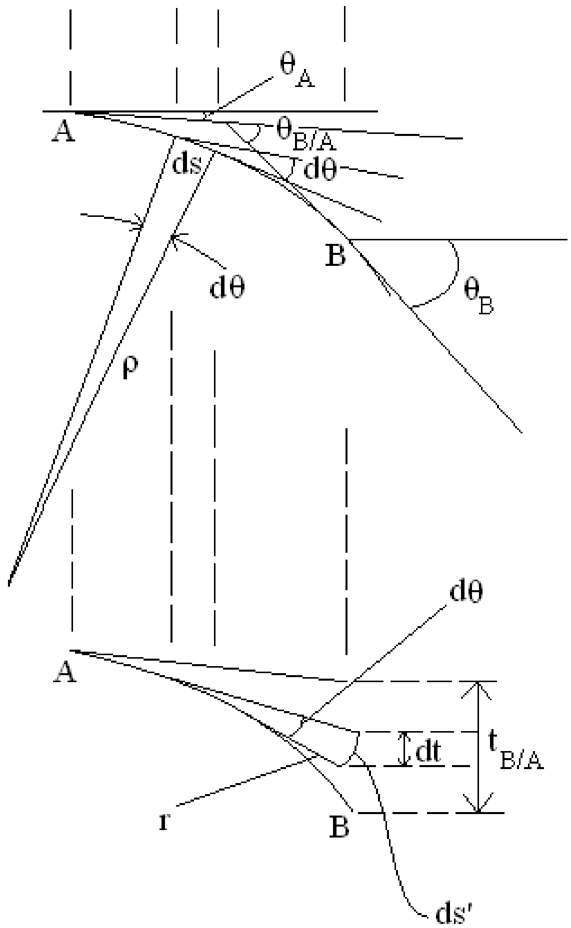


$$\theta_{B/A} = \theta_B - \theta_A$$

$$\frac{d\theta}{ds} \approx \frac{d\theta}{dx} = \frac{M}{EI} \Rightarrow d\theta = \frac{Mdx}{EI}$$

$$\theta_{B/A} = \int_A^B d\theta$$

$$\boldsymbol{\theta_B - \theta_A = \int_A^B \frac{Mdx}{EI}} \quad \text{(cantilevered beam)}$$



$t_{B/A} = t_B - t_A$

$\dfrac{d\theta}{2\pi}(2\pi r) = ds'$

$ds' = r\,d\theta$

$ds' \approx dt$ and $r \approx x_1$

$dt = x_1 d\theta = x_1 \dfrac{Mdx}{EI}$

(x_1 with respect to B as shown)

note: Although it may not be obvious, the assumptions made here are the same as $ds \approx dx$ and $\tan\theta \approx \theta$.

$t_{B/A} = \int_A^B dt$

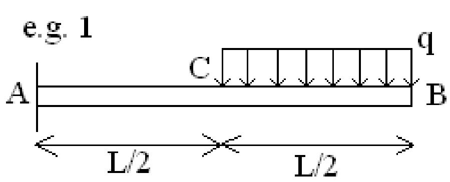
$$\mathbf{t_B - t_A = \int_A^B \frac{x_1 M dx}{EI}} \quad \text{(cantilevered beam)}$$

If A = fixed end, then $t_B = \delta_B$,

$t_A = (x_B - x_A)\theta_A = 0$ (cantilevered)

Note: For simply supported beams, since the concavity is reversed compared to cantilevered beams, θ is oriented differently, and t is on the opposite side of the deflection curve from δ. (see second e.x.)

e.g. 1
Find: δ_B



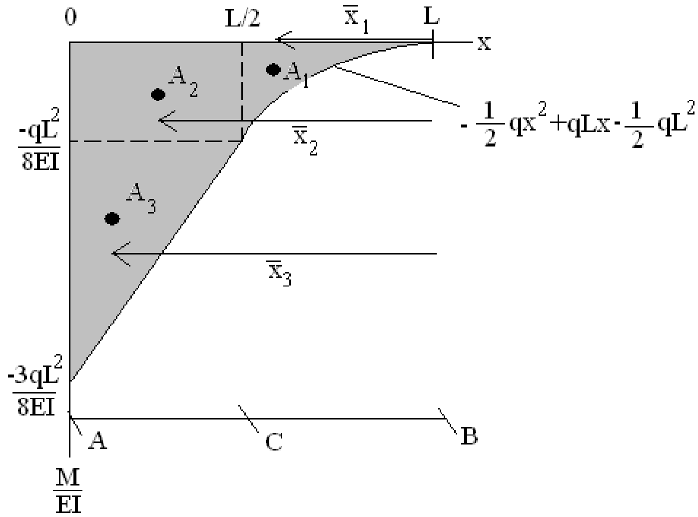
Method 1: find M(x) and solve $EI\upsilon''$

Method 2: find –q(x) and try to solve $EI\upsilon'''' \ \varepsilon\ [C, B]$

<u>Method 3: point load end point deflection formula and superposition</u>

Method 4: point load deflection formula for cantilevered beam for $a \le x \le L$ *and superposition*

Method 5: use area under $\dfrac{M}{EI}$ *diagram*



$$\overline{x_2}A_2 = \frac{L}{2}\left(\frac{qL^2}{8EI}\right)\left(\frac{L}{2}+\frac{L}{4}\right) = \frac{qL^3}{16EI}\left(\frac{3L}{4}\right)$$

$$\overline{x_3}A_3 = \frac{1}{2}\left(\frac{L}{2}\right)\left(\frac{3qL^2}{8EI}-\frac{qL^2}{8EI}\right)\left[\frac{L}{2}+\frac{2}{3}\left(\frac{L}{2}\right)\right]$$

$$= \frac{qL^3}{16EI}\left(\frac{5L}{6}\right)$$

$$\int_A^C dt = -\left[\frac{qL^3}{16EI}\left(\frac{3L}{4}\right)+\frac{qL^3}{16EI}\left(\frac{5L}{6}\right)\right]$$

$$\int_C^B dt = \int_C^B \frac{x_1 M}{EI}dx \quad where \;\; x_1 = L-x$$

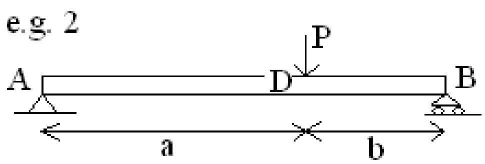
$$and \quad M = -\frac{1}{2}qx^2 + qLx - \frac{1}{2}qL^2$$

$$\Rightarrow \int_C^B \frac{x_1 M}{EI}dx = \int_{L/2}^{L} \frac{(L-x)(-\frac{1}{2}qx^2+qLx-\frac{1}{2}qL^2)}{EI}dx = \frac{-qL^4}{128EI}$$

$\delta_B - \delta_A = \delta_B$ *since* $\delta_A = 0$.

$$\Rightarrow \delta_B = \int_A^B dt = \frac{qL^3}{16EI}\left(\frac{3L}{4}\right)+\frac{qL^3}{16EI}\left(\frac{5L}{6}\right)+\frac{qL^4}{128EI} = \boldsymbol{\frac{41qL^4}{384EI}}$$

e.g. 2
Find: δ_D

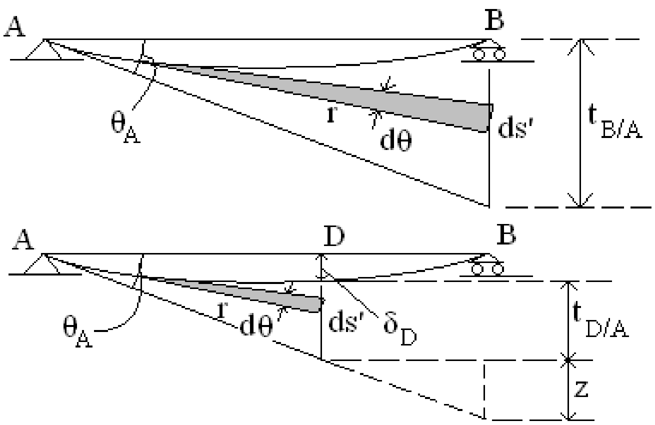


$ds' = r\,d\theta$

$ds' \approx dt \quad and \quad r \approx x_1$

Just as in the derivation for the cantilevered beam.

$$t_{B/A} = A_1\bar{x}_1 = \frac{Pab}{2EI}\left(\frac{L+b}{3}\right) = \frac{Pab}{6EI}(L+b)$$



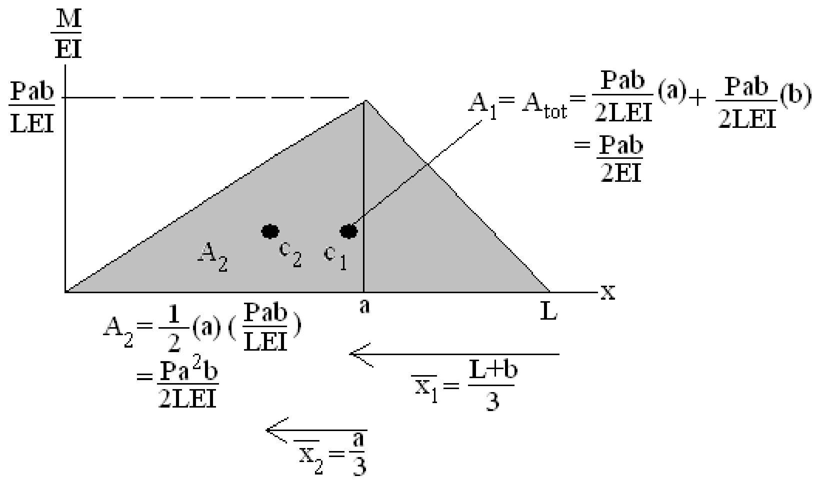
From similar triangles,

$$\frac{t_{B/A}}{L} = \frac{Pab}{6LEI}(L+b) = \frac{z}{b} \quad \left(\theta_A = \frac{t_{B/A}}{L}\right)$$

$$\Rightarrow z = \frac{Pab^2}{6LEI}(L+b)$$

$$t_{D/A} = A_2\bar{x}_2 = \left(\frac{Pa^2b}{2LEI}\right)\left(\frac{a}{3}\right) = \frac{Pa^3b}{6LEI}$$

$$\delta_D = t_{B/A} - (z + t_{D/A}) = \frac{Pa^2b^2}{3LEI}$$



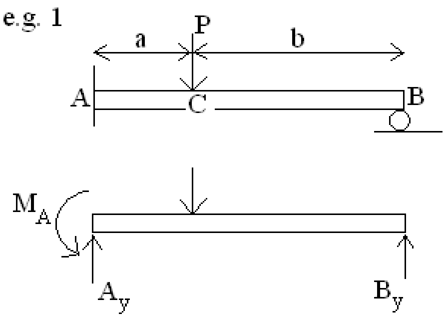
note: In either of these last two examples, a general formula for δ <u>would</u> have been possible using the moment-area method.

Simple statically indeterminate system (bending)

For a vertically-loaded beam, there are two relevant equations of equilibrium. If there are more than two unknowns (redundant supports) , then the extra equations needed may come from υ, υ', υ'', υ''' in terms of the unknown reactions. υ, υ', υ'', υ''' combined with the initial conditions, are the equations of compatibility.

e.g. 1
Find: M_A, A_y, B_y *for the propped cantilevered beam.*

e.g. 1



Equations of Equilibrium:

$$+\uparrow \sum F_y : A_y + B_y - P = 0 \Rightarrow B_y = P - A_y \quad (1)$$

$$+ \curvearrowleft \sum M_B : -A_y(L) + M_A + Pb = 0$$

$$\Rightarrow M_A = A_y L - Pb \quad (2)$$

$0 \le x \le a^- :$

$EI\upsilon''' = A_y$

$a^+ \le x \le L^- :$

$EI\upsilon_2''' = A_y - P$

$$EI\upsilon_1'' = -M_A + A_y x$$

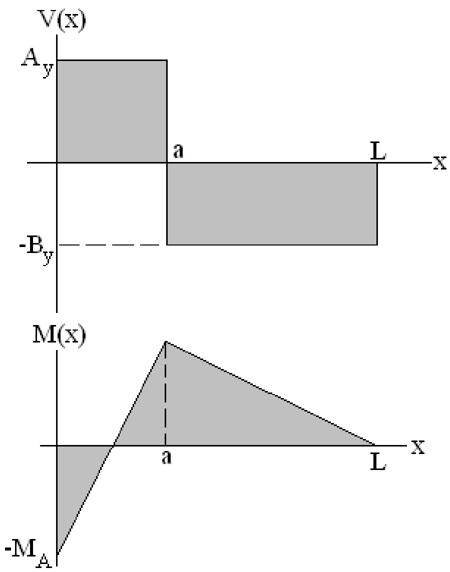
$$EI\upsilon_2'' = -M_A + A_y a + \int_a^x A_y - P dx$$
$$= -M_A + A_y a + A_y x - Px - A_y a + Pa$$

$$EI\upsilon_1' = -M_A x + \frac{A_y x^2}{2} + c_1$$

$$EI\upsilon_1 = \frac{-M_A x^2}{2} + \frac{A_y x^3}{6} + c_1 x + c_2$$

$$EI\upsilon_2' = -M_A x + \frac{A_y x^2}{2} - \frac{Px^2}{2} + Pax + d_1$$

$$EI\upsilon_2 = \frac{-M_A x^2}{2} + \frac{A_y x^3}{6} - \frac{Px^3}{6} + \frac{Pax^2}{2}$$
$$+ d_1 x + d_2$$



Boundary conditions:

$\upsilon_1(0) = 0 \Rightarrow c_2 = 0$

$\upsilon_1'(0) = 0 \Rightarrow c_1 = 0$

Continuity condition:

$\upsilon_1'(a) = \upsilon_2'(a) \Rightarrow d_1 = \dfrac{-Pa^2}{2}$ *(substituted (2))*

Boundary condition:

$\upsilon_2(L) = 0 \Rightarrow d_2 = \dfrac{1}{6}[3a^2 LP + 2L^3(A_y - P)]$

Extra continuity condition :
(equation of compatibility)

$\upsilon_1(a) = \upsilon_2(a) \Rightarrow$

$$0 = -\frac{Pa^3}{6} + \frac{Pa^3}{2} - \frac{Pa^3}{2} + \frac{1}{6}[3a^2 LP + 2L^3(A_y - P)] \qquad (3)$$

 # unknowns: $A_y, M_A, B_y = 3$

 # equations: 2 equil + 1 extra / compatibility = 3

$$\boldsymbol{A_y = \frac{Pb(3L^2 - b^2)}{2L^3} \quad B_y = \frac{Pa^2(3L - a)}{2L^3} \quad M_A = \frac{Pab(L+b)}{2L^2}}$$

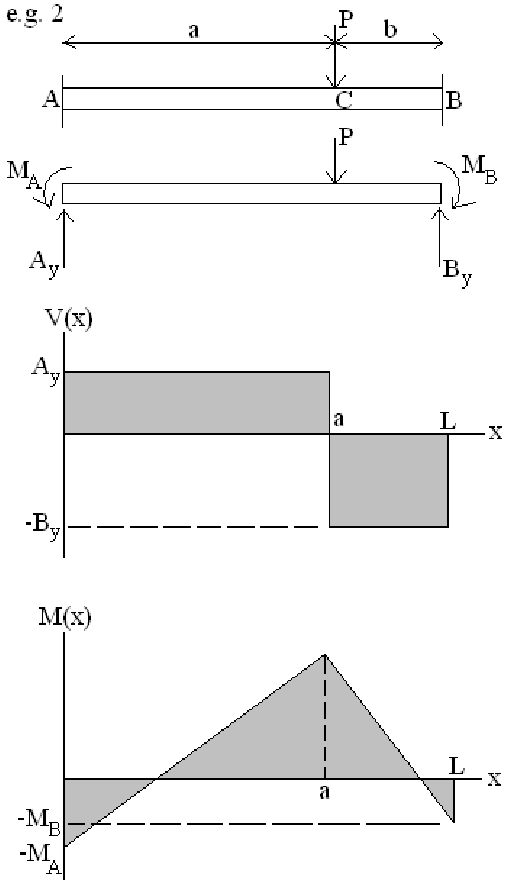
- *we can now find* σ, τ, *or* υ *like any statically determinate beam*

note: we could also find υ *for fixed A, load P, and* <u>no</u> *roller B, then find* υ *for fixed A, force* B_y*, and* <u>no</u> *load P, and* $\sum \upsilon(B) = 0$*, as our compatibility equation.*

e.g. 2

Find: A_y, B_y, M_A, M_B *for the fixed-end beam.*

e.g. 2



- Same exact differential equations as the previous e.x.-

But, now we have four unknowns.

The extra equation comes from $\upsilon_2'(L) = 0 \Rightarrow$
4 equations

$$+\uparrow \sum F_y : A_y + B_y - P = 0 \Rightarrow B_y = P - A_y \qquad (1)$$

$$\circlearrowleft + \sum M_B : -A_y(L) + M_A + Pb - M_B = 0$$
$$\Rightarrow M_B = M_A - A_y L + Pb \qquad (2)$$

note: $\upsilon_2'''(L) \neq 0$,
$\upsilon_2''(L) \neq 0$ *because the equation is valid for* $a \leq x \leq L^-$ *only.*

Boundary conditions:

$$\upsilon_1(0) = 0 \Rightarrow c_2 = 0$$
$$\upsilon_1'(0) = 0 \Rightarrow c_1 = 0$$
$$\upsilon_2'(L) = 0 \Rightarrow d_1 = \frac{1}{2} L[2M_A - 2aP + L(P - A_y)]$$
$$\upsilon_2(L) = 0 \Rightarrow d_2 = \frac{1}{6} L^2 [3aP - 3M_A + 2L(A_y - P)]$$

Extra conditions :
(equations of compatibility)

$$\upsilon_1(a) = \upsilon_2(a) \Rightarrow$$
$$\frac{1}{6} a^2 (A_y a - 3M_A) = \frac{1}{6}(a - L)^2 [2L(A_y - P) - 3M_A + a(2P + A_y)] \qquad (3)$$

$$\upsilon_1'(a) = \upsilon_2'(a) \Rightarrow$$
$$\frac{1}{2}[L^2(A_y - P) - a^2 P - 2L(M_A - aP)] = 0 \qquad (4)$$

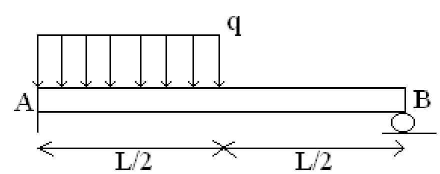
 # unknowns: $A_y, M_A, B_y, M_B = 4$
 # equations: 2 equil + 2 extra / compatibility = 4

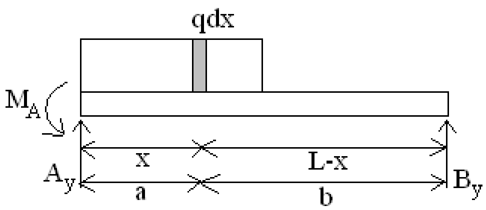
$$M_A = \frac{Pab^2}{L^2} \quad A_y = \frac{Pb^2}{L^3}(L+2a) \quad B_y = \frac{Pa^2}{L^3}(L+2b) \quad M_B = \frac{Pa^2 b}{L^2}$$

Superposition

e.g.
Find: A_y, B_y, M_A

e.g.





We can find the propped cantilevered reactions from a point load from a table or from the e.x. in the previous section.

$$A_y = \int_{L/2}^{L} \frac{qdb(b)(3L^2 - b^2)}{2L^3} = \int_{0}^{L/2} \frac{qdx(L-x)(3L^2 - (L-x)^2)}{2L^3}$$

$$= \boldsymbol{\frac{57qL}{128}}$$

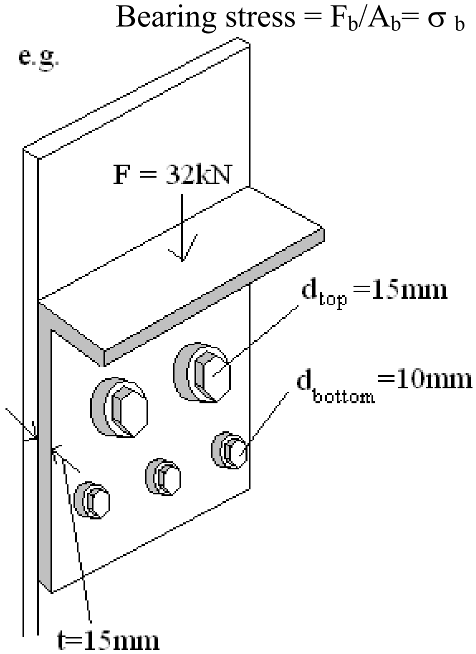
$$B_y = \int_{0}^{L/2} \frac{qda(a)^2(3L-a)}{2L^3} = \int_{0}^{L/2} \frac{qdx(x)^2(3L-x)}{2L^3} = \boldsymbol{\frac{7qL}{128}}$$

$$M_A = \int \frac{Pab[L+b]}{2L^2} = \int_{0}^{L/2} \frac{qdx(x)(L-x)[L+(L-x)]}{2L^2} = \boldsymbol{\frac{9qL^2}{128}}$$

Bearing and shear stress for connections

Bearing stress = $F_b/A_b = \sigma_b$ where A_b could be the thickness of a bolt plate multiplied by the diameter of the bolt, and F_b is the force acting on that bolt.

e.g.



Shear stress = V/A = τ where V is the shear force and A could be the cross-sect area of a bolt at the location of the bolt plate.

e.g.
Given: Force of 32kN on the angle bracket shown.
Find: σ_b *and* τ *for each bolt. (see below)*

$$(\sigma_b)_{top\ bolt} = (\frac{(A_b)_{top}}{total\ A_b})\frac{F}{(A_b)_{top}}$$

$$= \frac{32x10^3}{2(.015)(.015)+3(.015)(.01)} = 35.55MPa$$

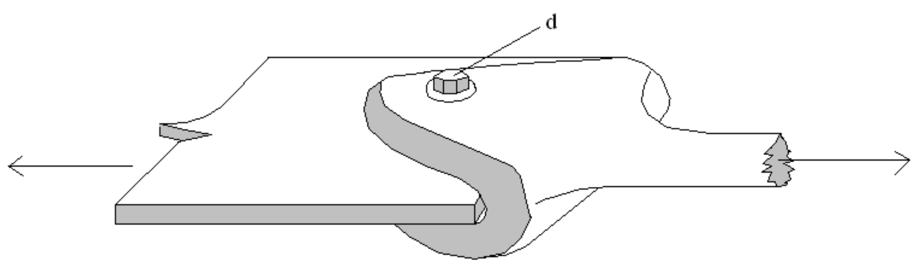
$$(\sigma_b)_{bottom\ bolt} = (\frac{(A_b)_{bottom}}{total\ A_b})\frac{F}{(A_b)_{bottom}} = 35.55MPA$$

$$(\tau)_{top\ bolt} = (\frac{A_{top}}{total\ A})\frac{F}{A_{top}}$$

$$= \frac{32x10^3}{\pi/4[2(.015)^2+3(.01)^2]} = 54.32MPa$$

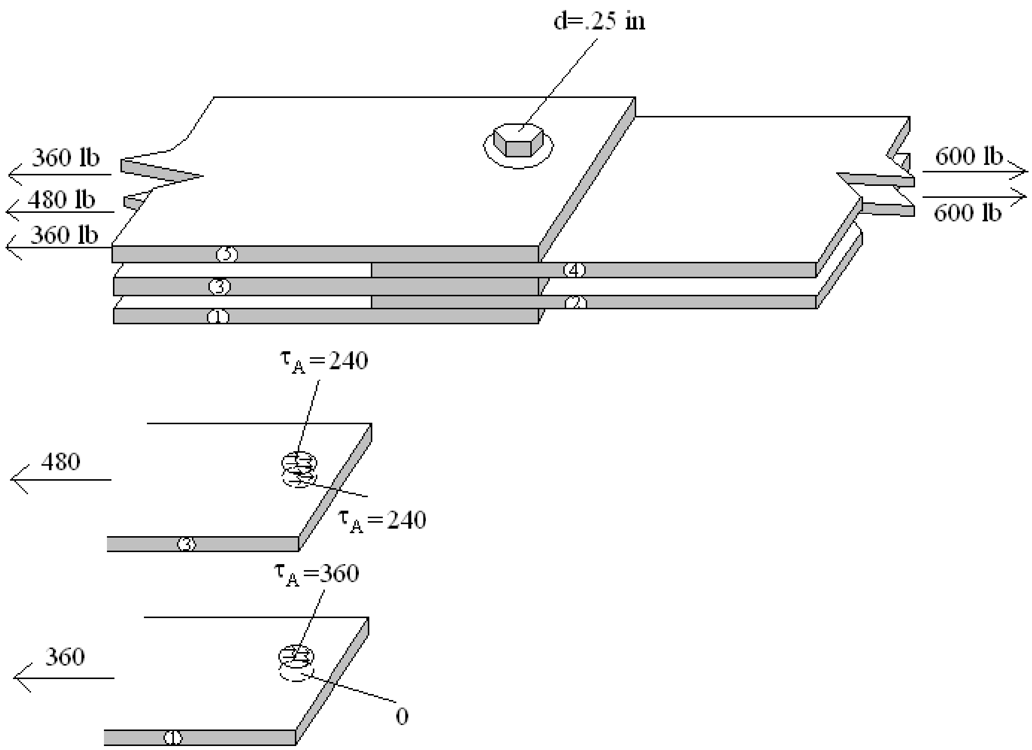
$$(\tau)_{bottom\ bolt} = (\frac{A_{bottom}}{total\ A})\frac{F}{A_{bottom}} = 54.32MPa$$

Double Shear



$$\tau = \frac{P/2}{\pi/4 (d)^2}$$

Multiple Shear



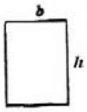
Strong and weak neutral axes, shear stresses, and bending stresses, for asymmetric sections, can be found in graduate level courses on Advanced Mechanics of Materials, Advanced Structural Analysis, or Advanced Steel Design. Buckling is another important phenomenon that is left to more advanced courses.

Works Cited

Gere, James M. Mechanics of Materials: Sixth Edition. Brooks/Cole. Belmont, CA 2004.

Lee, Vincent. Lecturer. University of Southern California. CE225. Spring 2005.

Appendix



Appendix A

SECTION PROPERTIES FOR SAWN LUMBER AND TIMBER

Nominal Size $b \times h$ in.	Standard Dressed Size (S4S) $b \times h$ in.	Area of Section A in.^{2}	X-X Axis: Moment of Inertia I in.^{4}	X-X Axis: Section Modulus S in.^{3}	Y-Y Axis: Moment of Inertia I in.^{4}	Y-Y Axis: Section Modulus S in.^{3}	Board Measure per Lineal Foot	Weight in pounds per linear foot of piece when weight of wood per cubic foot equals: 25 pcf	30 pcf	35 pcf
1×3	¾×2½	1.875	0.977	0.781	0.088	0.234	¼	0.326	0.391	0.456
1×4	¾×3½	2.625	2.680	1.531	0.123	0.328	⅓	0.456	0.547	0.638
1×6	¾×5½	4.125	10.398	3.781	0.193	0.516	½	0.716	0.859	1.003
1×8	¾×7¼	5.438	23.817	6.570	0.255	0.680	⅔	0.944	1.133	1.322
1×10	¾×9¼	6.938	49.466	10.695	0.325	0.867	⅚	1.204	1.445	1.686
1×12	¾×11¼	8.438	88.989	15.820	0.396	1.055	1	1.465	1.758	2.051
2×3*	1½×2½	3.750	1.953	1.563	0.703	0.938	½	0.651	0.781	0.911
2×4*	1½×3½	5.250	5.359	3.063	0.984	1.313	⅔	0.911	1.094	1.276
2×6*	1½×5½	8.250	20.797	7.563	1.547	2.063	1	1.432	1.719	2.005
2×8*	1½×7¼	10.875	47.635	13.141	2.039	2.719	1⅓	1.888	2.266	2.643
2×10*	1½×9¼	13.875	98.932	21.391	2.602	3.469	1⅔	2.409	2.891	3.372
2×12*	1½×11¼	16.875	177.979	31.641	3.164	4.219	2	2.930	3.516	4.102
2×14*	1½×13¼	19.875	290.775	43.891	3.727	4.969	2⅓	3.451	4.141	4.831
3×4	2½×3½	8.750	8.932	5.104	4.557	3.646	1	1.519	1.823	2.127
3×6	2½×5½	13.750	34.661	12.604	7.161	5.729	1½	2.387	2.865	3.342
3×8	2½×7¼	18.125	79.391	21.901	9.440	7.552	2	3.147	3.776	4.405
3×10	2½×9¼	23.125	164.886	35.651	12.044	9.635	2½	4.015	4.818	5.621
3×12	2½×11¼	28.125	296.631	52.734	14.648	11.719	3	4.883	5.859	6.836
3×14	2½×13¼	33.125	484.625	73.151	17.253	13.802	3½	5.751	6.901	8.051
3×16	2½×15¼	38.125	738.870	96.901	19.857	15.885	4	6.619	7.943	9.266
4×4	3½×3½	12.250	12.505	7.146	12.505	7.146	1⅓	2.127	2.552	2.977
4×6	3½×5½	19.250	48.526	17.646	19.651	11.229	2	3.342	4.010	4.679
4×8	3½×7¼	25.375	111.148	30.661	25.904	14.802	2⅔	4.405	5.286	6.168
4×10	3½×9¼	32.375	230.840	49.911	33.049	18.885	3⅓	5.621	6.745	7.869
4×12	3½×11¼	39.375	415.283	73.828	40.195	22.969	4	6.836	8.203	9.570
4×14	3½×13½	46.375	678.475	102.411	47.340	27.052	4⅔	8.047	9.657	11.266
4×16	3½×15½	53.375	1,034.418	135.661	54.487	31.135	5⅓	9.267	11.121	12.975
6×6	5½×5½	30.250	76.255	27.729	76.255	27.729	3	5.252	6.302	7.352
6×8	5½×7½	41.250	193.359	51.563	103.984	37.813	4	7.161	8.594	10.026
6×10	5½×9½	52.250	392.963	82.729	131.714	47.896	5	9.071	10.885	12.700
6×12	5½×11½	63.250	697.068	121.229	159.443	57.979	6	10.981	13.177	15.373
6×14	5½×13½	74.250	1,127.672	167.063	187.172	68.063	7	12.891	15.469	18.047
6×16	5½×15½	85.250	1,706.776	220.229	214.901	78.146	8	14.800	17.760	20.720
6×18	5½×17½	96.250	2,456.380	280.729	242.630	88.229	9	16.710	20.052	23.394
6×20	5½×19½	107.250	3,398.484	348.563	270.359	98.313	10	18.620	22.344	26.068
6×22	5½×21½	118.250	4,555.086	423.729	298.088	108.396	11	20.530	24.635	28.741
6×24	5½×23½	129.250	5,948.191	506.229	325.818	118.479	12	22.439	26.927	31.415
8×8	7½×7½	56.250	263.672	70.313	263.672	70.313	5⅓	9.766	11.719	13.672
8×10	7½×9½	71.250	535.859	112.813	333.984	89.063	6⅔	12.370	14.844	17.318
8×12	7½×11½	86.250	950.547	165.313	404.297	107.813	8	14.974	17.969	20.964
8×14	7½×13½	101.250	1,537.734	227.813	474.609	126.563	9⅓	17.578	21.094	24.609
8×16	7½×15½	116.250	2,327.422	300.313	544.922	143.313	10⅔	20.182	24.219	28.255
8×18	7½×17½	131.250	3,349.609	382.813	615.234	164.063	12	22.786	27.344	31.901
8×20	7½×19½	146.250	4,634.297	475.313	684.547	182.813	13⅓	25.391	30.469	35.547
8×22	7½×21½	161.250	6,211.484	577.813	755.859	201.563	14⅔	27.995	33.594	39.193
8×24	7½×23½	176.250	8,111.172	690.313	826.172	220.313	16	30.599	36.719	42.839
10×10	9½×9½	90.250	678.755	142.896	678.755	142.896	8⅓	15.668	18.802	21.936
10×12	9½×11½	109.250	1,204.026	209.396	821.651	172.979	10	18.967	22.760	26.554
10×14	9½×13½	128.250	1,947.797	288.563	964.547	203.063	11⅔	22.266	26.719	31.172
10×16	9½×15½	147.250	2,948.068	380.396	1,107.443	233.146	13⅓	25.564	30.677	35.790
10×18	9½×17½	166.250	4,242.836	484.896	1,250.338	263.229	15	28.863	34.635	40.408