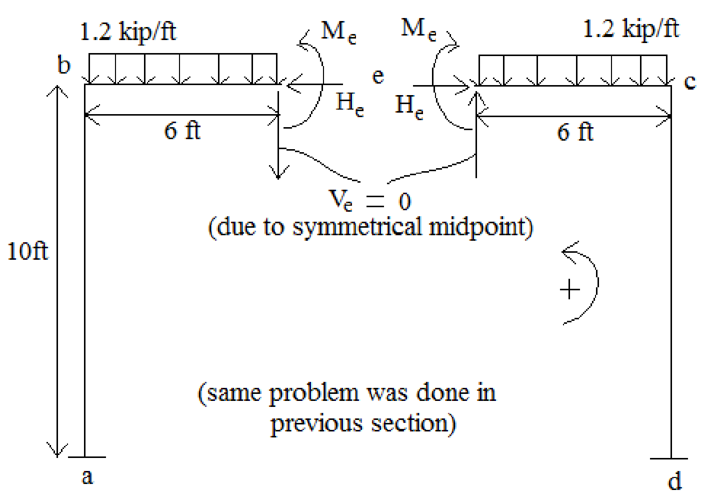
Castigliano's compatibility equation (method of least work) – Redundant forces

As mentioned at the beginning of the previous section, another way of finding redundants is through Castigiliano's Theorem:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} = \begin{bmatrix} \frac{dW}{dX_1} \\ \frac{dW}{dX_2} \\ \cdot \\ \cdot \\ \frac{dW}{dX_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

e.g.1

e.g. 1



Since e is a symmetrical midpoint,

$$\frac{dW}{dM_e} = 0 \text{ or } \int \frac{M(\frac{dM}{dM_e})}{EI} dx = 0$$

(slope at e = 0
)

Our other condition will be

$$\frac{dW}{dH_e} = 0 \text{ or } \int \frac{M(\frac{dM}{dH_e})}{EI} dx = 0$$

member	origin	limit	M	$\frac{dM}{dM_e}$	$\frac{dM}{dH_e}$
eb	e	0 to 6	$M_e - \frac{1.2x^2}{2}$	1	0
ba	b	0 to 10	$M_e - H_e x - 21.6$ (21.6 due to 1.2 kip/ft load)	1	x

"2" is in the numerator of the following expressions because we're only using half of the frame:

$$\frac{dW}{dM_e} = 0 \Rightarrow \frac{2}{EI}[\int_0^6 (M_e - \frac{1.2x^2}{2})(1)dx + \int_0^{10}(M_e + H_e x - 21.6)(1)dx] = 0$$

$$\Rightarrow .96M_e + 3H_e - 15.552 = 0 \qquad (1)$$

$$\frac{dW}{dH_e} = 0 \Rightarrow \frac{2}{EI}[\int_0^{10}(M_e + H_e x - 21.6)(x)dx] = 0$$

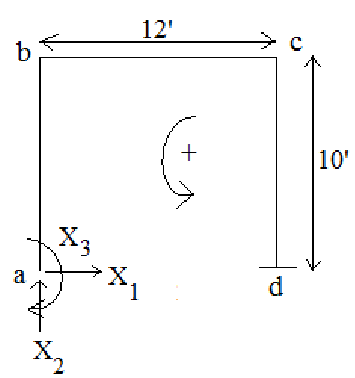
$$\Rightarrow 3M_e + 20H_e - 64.8 = 0 \qquad (2)$$

Solving (1) and (2) yields $H_e = 1.525 kips, M_e = 11.4353 kips * ft$

From equilibrium, $H_A = 1.53 kips; M_A = 5.08 kips$ *(skipped work)*
This is exact to two decimals. Calculations for this problem in the previous section were rounded, resulting in the slight difference in solution of M_A.

This example could just as easily of been solved by using the unit load method at the cut. Method of least work is still too cumbersome for a highly redundant frame.

e.g. 2 – Same a previous example, but apply the method of least work at "a" and take $\Delta_a = \begin{bmatrix} 7776 \\ -13478 \\ -1210 \end{bmatrix}$ *to be known.*



$M = M_{ab} + M_{bc} + M_{cd}$

$$= [-X_3 + X_1 x] + [-X_3 - X_2 x + 10X_1] + [-X_3 - 12X_2 + (10X_1 - X_1 x)]$$

$$\Delta_1 = \frac{1}{EI}[\int_0^{10}(-X_3 + X_1 x)(\frac{dM}{dX_1} = x)dx + \int_0^{12}(-X_3 - X_2 x + 10X_1)(10)dx$$
$$+ \int_0^{10}(-X_3 - 12X_2 + 10X_1 - X_1 x)(10 - x)dx] = 1867X_1 - 1320X_2 - 220X_3 \frac{kip * ft^3}{EI}$$

$$\Delta_2 = \frac{1}{EI}[\int_0^{10}(-X_3 + X_1 x)(\frac{dM}{dX_2} = 0)dx + \int_0^{12}(-X_3 - X_2 x + 10X_1)(-x)dx$$
$$+ \int_0^{10}(-X_3 - 12X_2 + 10X_1 - X_1 x)(-12)dx] = -1320X_1 + 2016X_2 + 192X_3 \frac{kip * ft^3}{EI}$$

$$\Delta_3 = \frac{1}{EI}\left[\int_0^{10}(-X_3 + X_1 x)\left(\frac{dM}{dX_3} = -1\right)dx + \int_0^{12}(-X_3 - X_2 x + 10X_1)(-1)dx\right.$$

$$\left. + \int_0^{10}(-X_3 - 12X_2 + 10X_1 - X_1 x)(-1)dx\right] = -220X_1 + 192X_2 + 32X_3\,\frac{kip*ft^2}{EI}$$

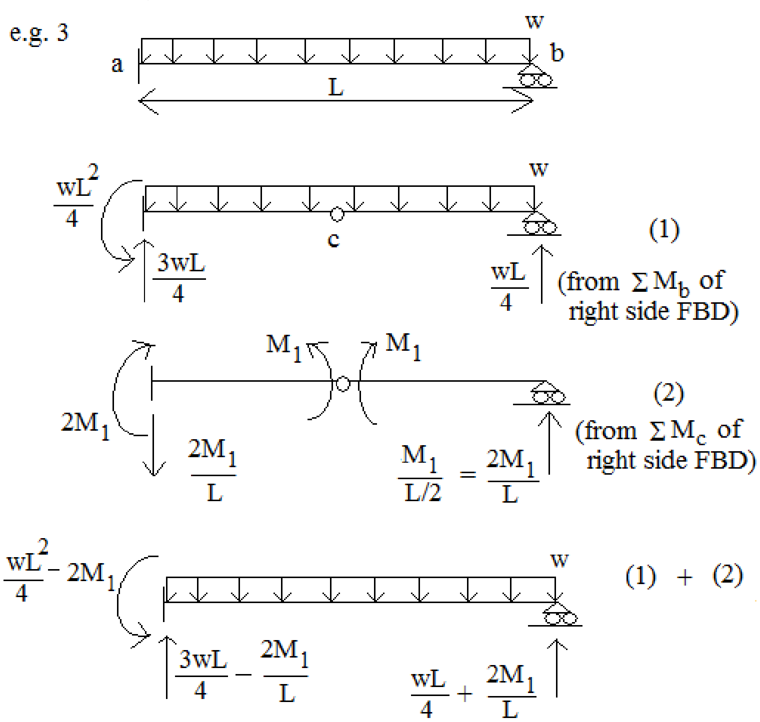
$$\begin{bmatrix} 7776 \\ -13478 \\ -1210 \end{bmatrix} + \begin{bmatrix} 1867X_1 - 1320X_2 - 220X_3 \\ -1320X_1 + 2016X_2 + 192X_3 \\ -220X_1 + 192X_2 + 32X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.53 \\ 7.20 \\ 5.16 \end{bmatrix} \begin{matrix} \text{kips} \\ \text{kips} \\ \text{kip*ft} \end{matrix}$$

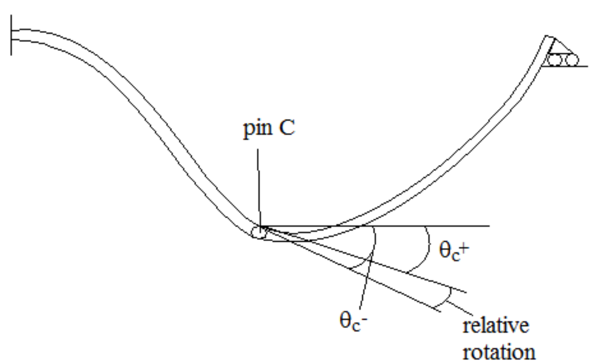
note: Δ_a did not have to be worked out separately – the moments M = 0, M = $\frac{1.2x^2}{2}$, and M = 86.4 could have been added to each respective integrand. The end result would be the same.

note: regardless of what deflections you choose to separate, or whether you use the unit load method or Castigliano's Theorem, separating the moments in a table, before integrating, is often a good idea.

e.g.3

reactions on the left in (1) and (2) are found from equilibrium of the whole beam after $\frac{wL}{4}$ and $\frac{2M_1}{L}$ are known, respectively.





Our redundant unknown is a <u>ccw</u> moment couple just to the left of the pin, which decreases the relative rotation (makes θ_c^- less steep) and an equal <u>cw</u> moment couple just to the right of the pin, which decreases the relative rotation (makes θ_c^+ more steep).

So, we call these moments M_1 as shown in (2). In other words, we defined an M_1 that causes some <u>*rotation*</u> *($\theta_c^- - \theta_c^+$ to cancel the effect of the pin). In the real beam, this* <u>*rotation*</u> *is zero. So,* $\frac{dW}{dM_1} =$ <u>*rotation*</u> *from* $M_1 = 0$.

$x = 0^+ :$

$$V(0^+) = \frac{3wL}{4} - \frac{2M_1}{L} \qquad M(0^+) = \frac{-wL^2}{4} + 2M_1$$

$0^+ \le x \le L^- :$

$$V(x) = \frac{3wL}{4} - \frac{2M_1}{L} - \int_0^x w dx = \frac{3wL}{4} - \frac{2M_1}{L} - wx$$

$$M(x) = \frac{-wL^2}{4} + 2M_1 + \int_0^x \frac{3wL}{4} - \frac{2M_1}{L} - wx\, dx = \frac{-.5[Lw(x^2 - 1.5Lx + .5L^2) + 4M_1(x-L)]}{L}$$

(equivalent to finding the BMD for (1) and (2) separately and summing)

$x = L^- :$

$$V(L^-) = \frac{-wL}{4} - \frac{2M_1}{L} \qquad M(L^-) = 0$$

$$\frac{dW}{dM_1} = \frac{1}{EI}\int_0^L \left(\frac{-.5[Lw(x^2 - 1.5Lx + 4M_1(x-L)]}{L}\right)\left(\frac{-2(x-L)}{L}\right)dx = 0$$

$$\Rightarrow M_1 = \frac{wL^2}{16} \quad \therefore \quad R_b = \frac{wL}{4} + \frac{2(\frac{wL^2}{16})}{L} = \frac{3}{8}Lw$$

note: Axial deformation is neglected for beams and frames in the method of consistent deformations, least work, and other methods to come. But what effect does this have on the accuracy of the calculations of redundant forces?

<u>Works Cited</u>

Hsieh, Yuan-Yu, and S.T. Mau. <u>Elementary Theory of Structures: Fourth Edition</u>. Prentice Hall. Upper Saddle River, NJ 1995.

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