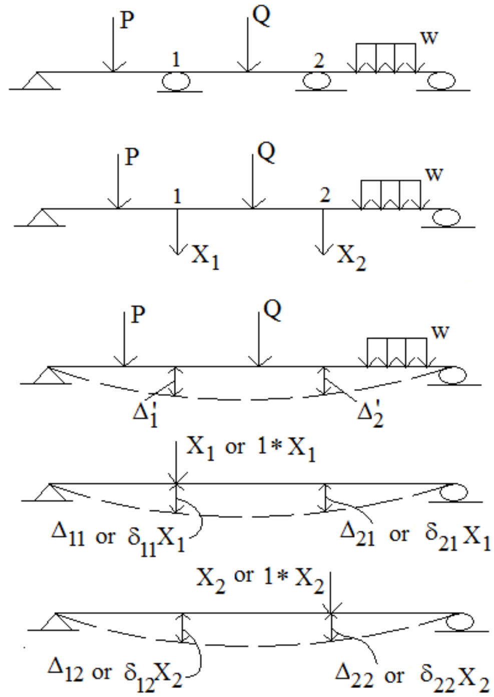
Method of consistent deformations – Redundant forces

$\Delta'_1 + \Delta_{11} + \Delta_{12} = 0$



These *could* be combined into a single integral (so that M is in terms if P, Q, X_1, and X_2). Then, it would just be Δ_1 (from all forces) = 0.

Specifically, $\Delta_1 = \int_0^L \frac{M(\frac{dM}{dX_1})}{EI} dx$ (if using Castigliano's Theorem)

Also,

$\Delta'_2 + \Delta_{21} + \Delta_{22} = 0$

or

$\Delta'_1 + \delta_{11} X_1 + \delta_{12} X_2 = 0$ (1)

$\Delta'_2 + \delta_{21} X_1 + \delta_{22} X_2 = 0$ (2)

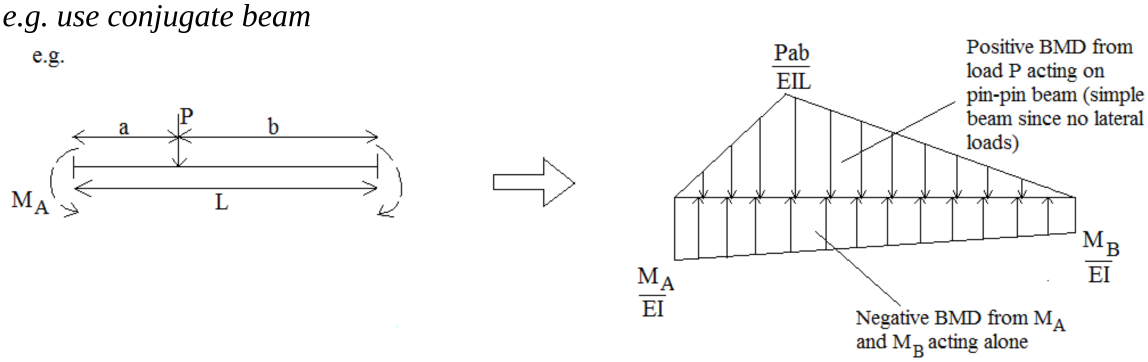
Δ = deflection due to external loads (with redundant supports removed).

δ_{11} = deflection at point 1 due to a unit force at point 1

δ_{12} = deflection at point 1 due to a unit force at point 2

This applies to couples and/or loads.
We can use Castigliano's Theorem, the unit load method, or any other method.
There are two unknowns X_1, X_2, and two equations (1), (2) $\Rightarrow$
Solve for the redundant supports. Then, find the rest of the support reactions.
(clearly X_1 and X_2 as pictured will have negative values)

e.g. use conjugate beam



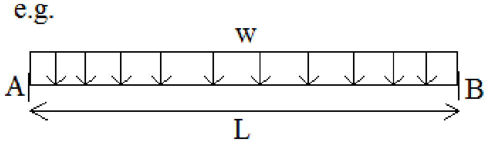
(conjugate beam)

From equilibrium of the loaded conjugate beam,

$$M_A = \frac{Pab^2}{L^2} \quad ; \quad M_B = \frac{Pa^2b}{L^2}$$

e.g. use superposition

e.g.



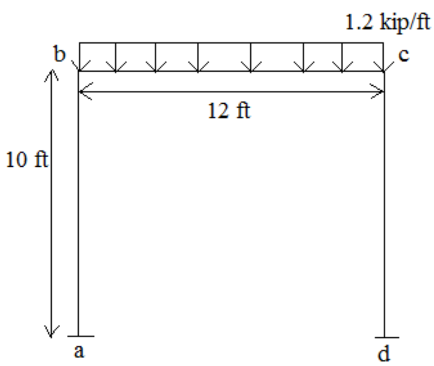
We know that $M_A = \frac{Pab^2}{L^2}$ *for an arbitrary point load (derived in the previous example).*
For this uniform load, if P = wdx, a = x, and b = L-x, then

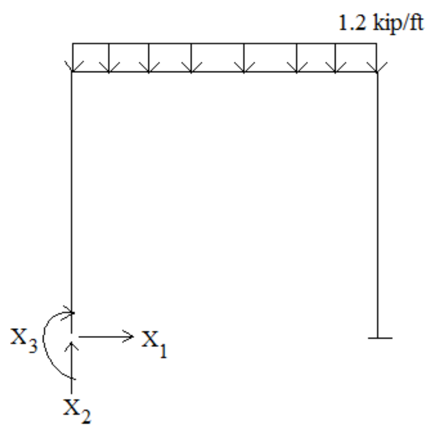
$$M_A = \int_0^L \frac{w\,dx(x)(L-x)^2}{L^2} = \frac{1}{12}L^2 w$$

When faced with fixed-end beams or propped-cantilevered beams, reactions can be determined by this approach regardless of load distribution, as long as we know the reactions for an arbitrary point load (M_A *for a propped-cantilevered beam from an arbitrary point load =* $\frac{Pab(L+b)}{2L^2}$*)*

e.g. use unit load method

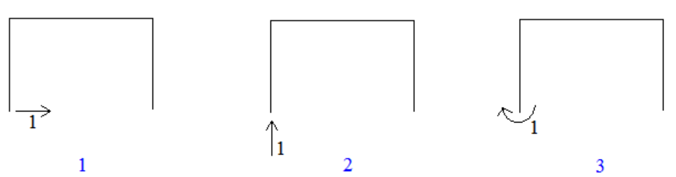
e.g.





We can approach this problem by using a system of equations such as (1) and (2) on the previous page. We've already found " Δ_1 *", "* Δ_2 *", and "* Δ_3 *" from e.g. 2 in the "virtual force (unit load method)" section*

$$\begin{bmatrix} \Delta'_1 \\ \Delta'_2 \\ \Delta'_3 \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



member	origin	m_1	m_2	m_3
ab	a	-x	0	1
bc	b	-10	x	1
cd	c	x-10	12	1

we need the deflections at "a" for the combined loading of the unit forces in 1, 2, 3

$$General: \quad \delta = \int_{ab} \frac{Mm}{EI} dx + \int_{bc} \frac{Mm}{EI} dx + \int_{cd} \frac{Mm}{EI} dx$$

δ_{11} : deflection in direction 1 due to 1

δ_{12} : deflection in direction 1 due to 2

δ_{13} : deflection in direction 1 due to 3

δ_{21} : deflection in direction 2 due to 1

δ_{22} : deflection in direction 2 due to 2

δ_{23} : deflection in direction 2 due to 3

δ_{31} : deflection in direction 3 due to 1

δ_{32} : deflection in direction 3 due to 2

δ_{33} : deflection in direction 3 due to 3

$(\delta_a)_1 =$ deflection at "a" in direction 1

$$= \delta_{11} + \delta_{12} + \delta_{13} = [\int_{ab} \frac{(m_1)^2}{EI} dx + \int_{bc} \frac{(m_1)^2}{EI} dx + \int_{cd} \frac{(m_1)^2}{EI} dx]$$

$$+ [\int_{ab} \frac{m_2 m_1}{EI} dx + \int_{bc} \frac{m_2 m_1}{EI} dx + \int_{cd} \frac{m_2 m_1}{EI} dx] + [\int_{ab} \frac{m_3 m_1}{EI} dx + \int_{bc} \frac{m_3 m_1}{EI} dx + \int_{cd} \frac{m_3 m_1}{EI} dx]$$

$(\delta_a)_1 =$ deflection at "a" in direction 2

$$= \delta_{21} + \delta_{22} + \delta_{23} = [\int_{ab} \frac{m_1 m_2}{EI} dx + \int_{bc} \frac{m_1 m_2}{EI} dx + \int_{cd} \frac{m_1 m_2}{EI} dx]$$

$$+ [\int_{ab} \frac{(m_2)^2}{EI} dx + \int_{bc} \frac{(m_2)^2}{EI} dx + \int_{cd} \frac{(m_2)^2}{EI} dx] + [\int_{ab} \frac{m_3 m_2}{EI} dx + \int_{bc} \frac{m_3 m_2}{EI} dx + \int_{cd} \frac{m_3 m_2}{EI} dx]$$

$(\delta_a)_1 =$ deflection at "a" in direction 3

$$= \delta_{31} + \delta_{32} + \delta_{33} = [\int_{ab} \frac{m_1 m_3}{EI} dx + \int_{bc} \frac{m_1 m_3}{EI} dx + \int_{cd} \frac{m_1 m_3}{EI} dx]$$

$$+ [\int_{ab} \frac{m_2 m_3}{EI} dx + \int_{bc} \frac{m_2 m_3}{EI} dx + \int_{cd} \frac{m_2 m_3}{EI} dx] + [\int_{ab} \frac{(m_3)^2}{EI} dx + \int_{bc} \frac{(m_3)^2}{EI} dx + \int_{cd} \frac{(m_3)^2}{EI} dx]$$

$$\delta_{11} = \frac{1}{EI} [\int_0^{10} (-x)^2 dx + \int_0^{12} (-10)(-10) dx + \int_0^{10} (x-10) dx] = 1867 \, {}^{kip * ft^3}\!/_{EI} \quad (right)$$

$$\delta_{12} = \frac{1}{EI} [0 + \int_0^{12} (-10)(x) dx + \int_0^{10} (x-10)(12) dx] = -1320 \, {}^{kip * ft^3}\!/_{EI} \quad (left)$$

$$\delta_{13} = \frac{1}{EI}[\int_0^{10}(-x)(1)dx + \int_0^{12}(-10)(1)dx + \int_0^{10}(x-10)(1)dx] = -220\, {kip*\ ft^3}/{EI} \quad (left)$$

$$\delta_{21} = \delta_{12} = -1320\, {kip*\ ft^3}/{EI} \quad (down)$$

$$\delta_{22} = \frac{1}{EI}[0 + \int_0^{12}x^2dx + \int_0^{10}12^2dx] = 2016\, {kip*\ ft^3}/{EI} \quad (up)$$

$$\delta_{23} = \frac{1}{EI}[0 + \int_0^{12}(x)(1)dx + \int_0^{10}(12)(1)dx] = 192\, {kip*\ ft^3}/{EI} \quad (up)$$

$$\delta_{31} = \delta_{13} = -220\, {kip*\ ft^2}/{EI} \quad (ccw)$$

$$\delta_{32} = \delta_{23} = 192\, {kip*\ ft^2}/{EI} \quad (cw)$$

$$\delta_{33} = \frac{1}{EI}[\int_0^{10}dx + \int_0^{12}dx + \int_0^{10}dx] = 32\, {kip*\ ft^2}/{EI} \quad (cw)$$

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1867 & -1320 & -220 \\ -1320 & 2016 & 192 \\ -220 & 192 & 32 \end{bmatrix}$$

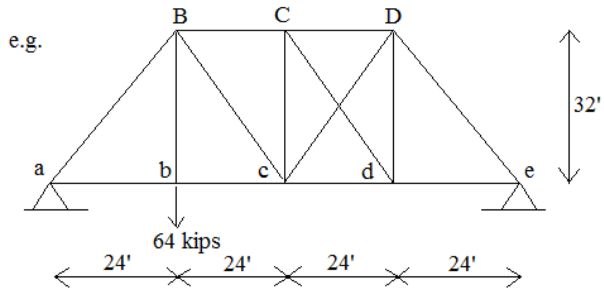
$$\begin{bmatrix} 7776 \\ -13478 \\ -1210 \end{bmatrix} + \begin{bmatrix} 1867 & -1320 & -220 \\ -1320 & 2016 & 192 \\ -220 & 192 & 32 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.53 \\ 7.20 \\ 5.16 \end{bmatrix} \begin{matrix} \text{kips} \\ \text{kips} \\ \text{kip*ft} \end{matrix}$$

note: $(1.2\frac{kips}{ft})(12ft) = 14.4$ *kips ; From symmetry,* $\frac{14.4}{2} = 7.2 kips$

note: Making a table greatly simplified this problem. Separating all of the deflections and summing is not necessary, but was done for clarity. The end result would be the same.

Using the method of consistent deformations in analyzing a frame would become intolerable if the problem involves as many redundant elements as a rigid frame usually does.

e.g.



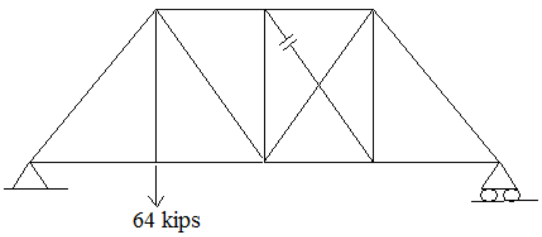
$E = 30{,}000\ {}^{kips}\!/_{in^2}$

$\dfrac{L(\,ft\,)}{A(\,in^2\,)} = 1$ *for all members*

note: $14 + 4 > 2j = 16$

(redundant to the 2^{nd} *degree)*

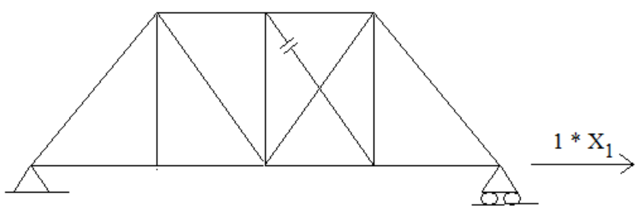
Two redundant elements; one in the reaction component (choose "e") and the other in the bar (choose Cd).



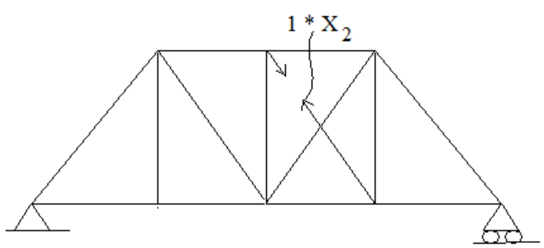
The horizontal movement at support e and the relative axial displacement between cut ends of bar Cd are zero.

One way to think of it is:

Δ_2 *and* δ_{21} *cause joints C and d to move* *<u>closer to each other</u>* *along the line Cd. The cut ends overlap.*



For beam Cd to be one piece, its unknown internal force X_2*, must shorten the beam by* δ_{22} *so that the cut ends no longer overlap. The end result is a shorter beam Cd, but no displacement between cuts.*



member	S'	u_1	u_2	$\frac{S'u_1L}{A}$	$\frac{S'u_2L}{A}$	$\frac{u_1^2L}{A}$	$\frac{u_2^2L}{A}$	$\frac{u_1u_2L}{A}$	$S = S' + u_1X_1 + u_2X_2$
ab	36	1	0	36	0	1	0	0	36 - 25.6 + 0 = 10.4
bc	36	1	0	36	0	1	0	0	36 - 25.6 + 0 = 10.4
cd	12	1	-3/5	12	-7.2	1	9/25	-3/5	12 - 25.6 + 6.4 = -7
de	12	1	0	12	0	1	0	0	12 - 25.6 + 0 = -13.6
BC	-24	0	0	0	0	0	0	0	-24 + 0 + 0 = -24
CD	-24	0	-3/5	0	14.4	0	9/25	0	-24 + 0 + 6.4 = -17.6
aB	-60	0	0	0	0	0	0	0	-60 + 0 + 0 = -60
Bb	64	0	0	0	0	0	0	0	64 + 0 + 0 = 64
Bc	-20	0	0	0	0	0	0	0	-20 + 0 + 0 = -20
Cc	0	0	-4/5	0	0	0	16/25	0	-20 + 0 + 8.5 = 8.5
Cd	0	0	1	0	0	0	1	0	0 + 0 - 10.6 = -10.6
cD	20	0	1	0	20	0	1	0	20 + 0 - 10.6 = 9.4
Dd	0	0	-4/5	0	0	0	16/25	0	0 + 0 + 8.5 = 8.5
De	-20	0	0	0	0	0	0	0	-20 + 0 + 0 = -20
			Σ	96	27.2	4	4	-3/5	
				Δ_1	Δ_2	δ_{11}	δ_{22}	$\delta_{12} = \delta_{21}$	

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Sigma \frac{S'u_1L}{AE} \\ \Sigma \frac{S'u_2L}{AE} \end{bmatrix} + \begin{bmatrix} \Sigma \frac{u_1^2 L}{AE} & \Sigma \frac{u_2 u_1 L}{AE} \\ \Sigma \frac{u_1 u_2 L}{AE} & \Sigma \frac{u_2^2 L}{AE} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 96 \\ 27.2 \end{bmatrix} + \begin{bmatrix} 4 & -3/5 \\ -3/5 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -25.6 \\ -10.6 \end{bmatrix} \text{kips}$$

member Cd is compressive

note: Deformation must always be considered when the truss is statically indeterminate. Using method of sections, for example, would not work because it would yield a singular solution.

Works Cited

Hsieh, Yuan-Yu, and S.T. Mau. Elementary Theory of Structures: Fourth Edition. Prentice Hall. Upper Saddle River, NJ 1995.

Trifunac, Mihailo. Lecturer. University of Southern California. CE358. Fall 2005.