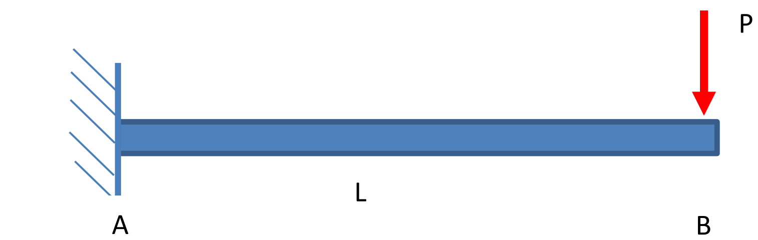
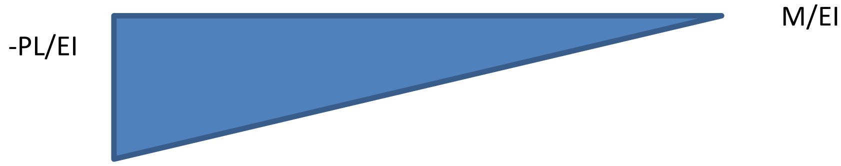
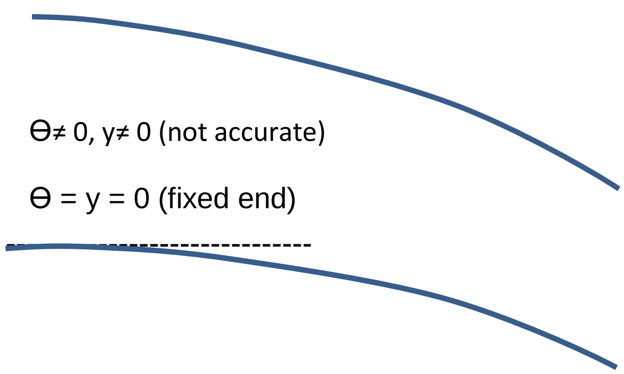
The conjugate beam method is a method that allows us **to compute the slopes and deflections in real beams by determining the shear and moment forces in a conjugate beam**, that is, a beam with altered boundary conditions. Because the conjugate beam is simply an extenuation of the elastic load method, it is necessary to first describe its predecessor. The elastic load method is simpler than the conjugate beam method in that **it only applies to simply supported beams**. This method was introduced as a way to avoid using complex tangential deviations and changes in slope to determine slope and deflection, as well to avoid being forced to draw a perfect diagram of the deflected beam. Instead, shear and moment curves which many students are more comfortable and confident in, can be used to relate shear to slope, and moment to deflection along a beam. Firstly, a moment diagram is generated by the real loads on the beam; this diagram becomes an M/EI diagram which represents angle change per unit length, the angle change in the deformed shape. This diagram is then applied to the original beam as a load (aka elastic load), and the shear and moment at any point along this beam relate respectively to slope and deformation. The elastic load method is relatively simple conceptually because the supports that make up a simply supported beam function the same way, whereas the invention of the conjugate beam method was necessary because of the many supports that exist and that make relating shear and moment to slope and deformation difficult because of the boundary conditions they impose. Thus, the conjugate beam method consists of the same steps with one addition. Firstly, the moment diagram of the original beam with its original load is drawn. This becomes our M/EI diagram and subsequently the load which is imposed on our conjugate beam. The step that the elastic load method exempts, but which is integral to the conjugate beam method, is the swapping of supports. As we will see in a moment in greater detail, the conjugate beam's supports must be replaced with supports that impose the boundary conditions of the original beam was under, while under the M/EI load. To introduce the idea behind the conjugate beam method, we will observe an example featuring a cantilevered beam loaded at its free end by a force, P. As we go through examples, keep in mind these tips.[1][2][3]



Original cantilevered beam with point load at the end.

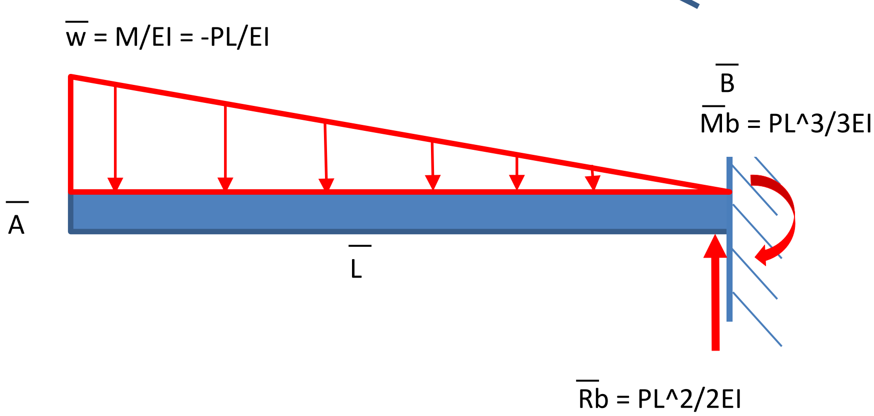


Moment diagram due to load P on original beam. Diagram becomes M/EI diagram.



Deflected shape that would occur from M/EI diagram. Notice that the slope and deflection at what is the fixed end is not zero. Thus, this shape is unrealistic because the left side is fixed.

Realistic shape that would occur because slope and deflection at end with fixed support would be zero.



M/EI diagram appears as loading on conjugate beam. To ensure slope and deflection on left end is zero, to match the actual deflected shape, the conjugate beam's fixed end becomes a free end. However, now a support must exist with shear and moment capacity. Thus the right end, originally free, becomes fixed.

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[1] Θ = slope, y = deflection, *V* = shear, M = Moment
[2] Variables under a line, such as $\overline{V}$, indicate that they are conjugate variables
[3] Note that when the M/EI diagram is positive, the $\overline{w}$ load acts in the positive direction

Derivation of Conjugate Beam Method

To get a better idea of how we arrived at the conjugate beam method, we will derive the compatibility equations that allow us to perform the calculations. Deriving the equations that prove the conjugate beam method is fairly straight forward. We just need to find a way to mathematically relate slope to shear, and deflection to moment. Note that **the rotation in the real beam is equal to the shear in the conjugate beam, and the deflection in the real beam is equal to the moment in the conjugate beam.** See Appendix A. for the conjugate supports which replace the real supports as you apply the method.

$$\overline{w} = \frac{M}{EI}$$

The load in our conjugate beam is equal to our M/EI, where M is the real bending moment in our real beam.

$$y' = \Theta = \int \frac{M}{EI} dx$$

The first derivative of our deflection is equal to our slope, which is also equal to the integral of our angle change per unit length. This makes sense because the slope is simply a rate of rise over run.

$$y = \iint \frac{M}{EI} dx$$

We know that deflection is equal to the double integral of M/EI, or the double integral of the angle change per unit length.

$$y' = \Theta = \iint \overline{w}\, dx = \overline{\mathrm{V}}$$

Our slope is also equal to the double integral of our load on the conjugate beam, which is equal to the shear in our conjugate beam.

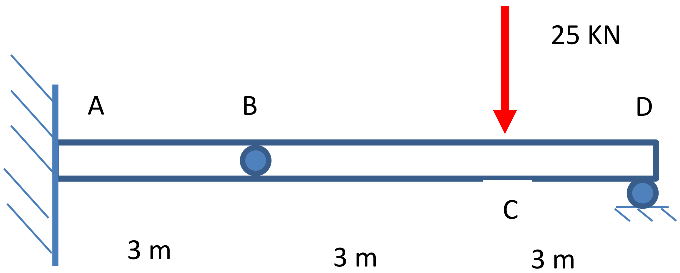
$$y = \int \overline{V}\, dx = \overline{M}$$

Our slope is also equal to the integral of the shear on our conjugate beam, which is equal to the moment in our conjugate beam.

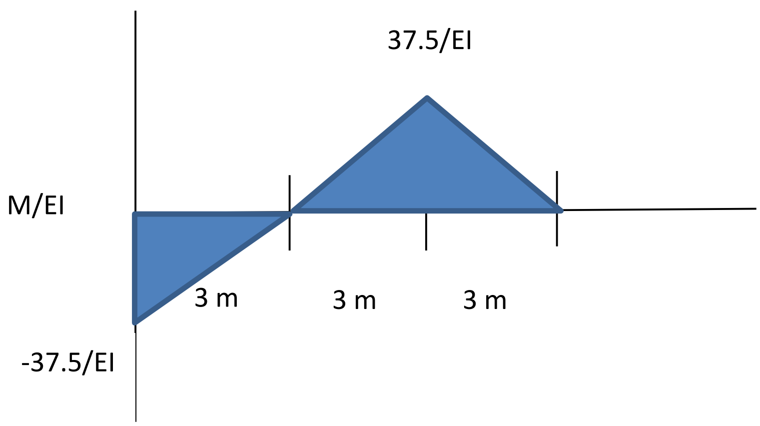
Simplifying so that we relate slope and deflection directly to the shear and moment in our conjugate beam give us the following equations, which allow us to use the conjugate beam method and determine the appropriate supports on our conjugate beam to replace those on the original, real beam.

$$\Theta = \overline{\mathrm{V}}$$

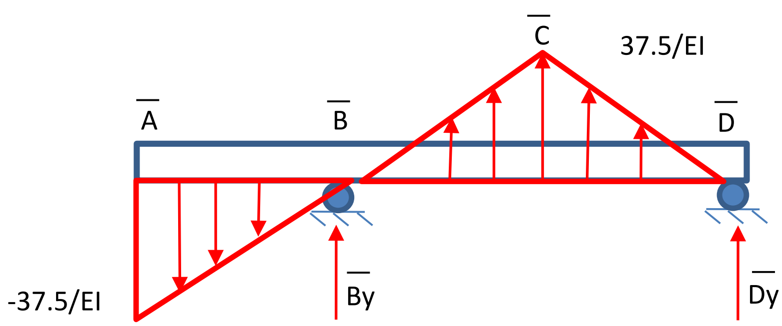
$$y = \overline{M}$$



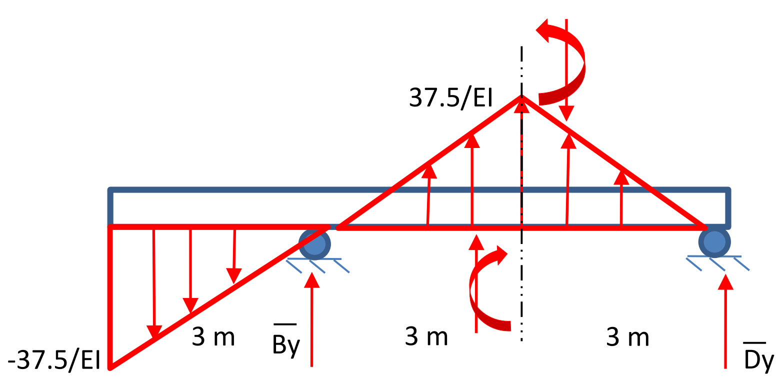
Original beam with fixed end, hinge, and roller at left end. Point load acts on beam. We will solve for the deflection at point C, yc.



Step 1: Draw Moment diagram, which becomes M/EI diagram. Use statics to find moment values



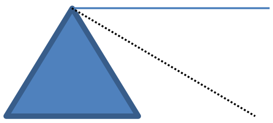
Step 2: Apply M/EI diagram as a load to conjugate beam. Using Appendix A, we know that the fixed end becomes a free end, the roller end stays the same, and the hinge becomes a roller. Treat the load as you would any other distributed load – solve using shapes or integrals.



Step 3: Use statics to determine $\overline{D}y$.

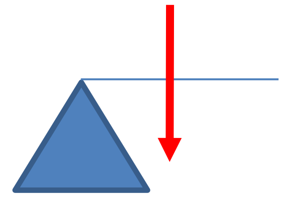
Lastly, cut the beam at $\overline{C}$ to solve for the moment, which is equal to the deflection, yc.

Appendix A.
Shows supports and conditions of real beams and their conjugate counterparts
Θ = slope, y = deflection, V = shear, M = Moment



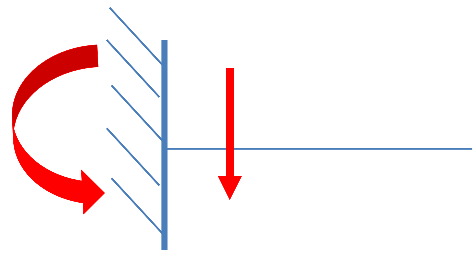
Roller and Pin
$\Theta \neq 0$
$y = 0$

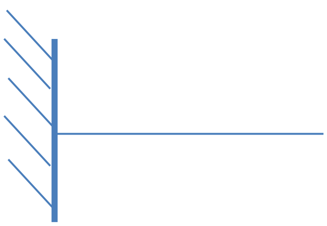
Roller and Pin
$\overline{V} \neq 0$
$\overline{M} = 0$



Free End
$\Theta \neq 0$
$y \neq 0$

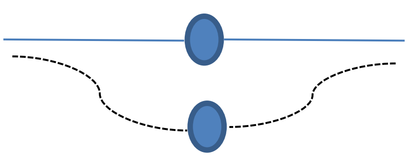
Fixed End
$\overline{V} \neq 0$
$\overline{M} \neq 0$





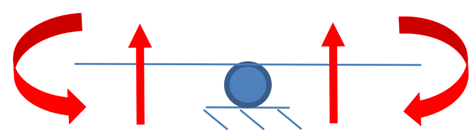
Fixed End
$\Theta = 0$
$y = 0$

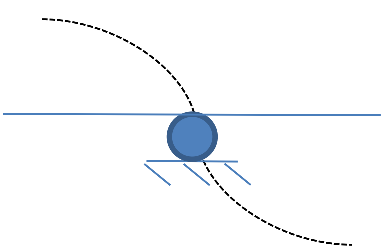
Free End
$\overline{V} = 0$
$\overline{M} = 0$



Hinge
$\Theta \neq 0$ & discontinuous
$y \neq 0$ & continuous

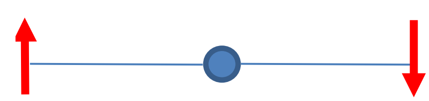
Roller (at loc of hinge)
$\overline{V} \neq 0$ & discontinuous
$\overline{M} \neq 0$ & continuous





Roller (not at end)
$\Theta \neq 0$ & continuous
$y = 0$

Hinge
$\overline{V} \neq 0$ & continuous
$\overline{M} = 0$



Works Cited

Kenneth M. Leet, “Fundamentals of Structural Analysis,” 4th ed. (McGraw-Hill: New York, NY 2008), pg 335, 339. Assistance to the author. Written help. This textbook gave me the explanations of the elastic beam method and the conjugate beam method. The first page of this paper, and its explanation of how both methods work, compare, and differ, are the result of paraphrasing this textbook. Additionally, the first example featuring the cantilevered beam are taken directly from pg. 339.