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CLASSICAL STRUCTURAL ANALYSIS

Classical structural analysis is concerned with methods for finding axial forces, shear forces, and internal moments, within structures, when the applied external forces are known. Whereas “statics” deals with the equations of static equilibrium, which can be used to find internal forces (axial, shear, and moments) for statically determinate structures, classical structural analysis can handle more complicated structures that are statically indeterminate. The equations of static equilibrium will still be important, but now we also need to consider the properties of the materials that the structure is composed of.

The Young’s Modulus of the material will be important, for example. We will also need to know something about the geometry of the cross-sections of members, such as the cross-sectional area and moment of inertia. Thus, “statics” as well as “mechanics of materials” are topics that are prerequisites for this topic: “classical structural analysis.” There are other kinds of structural analysis, such as “finite element analysis,” but such methods use algorithms that are better implemented using a computer. Classical structural analysis, on the other hand, has been around for a long time and is meant to be performed by hand. In other words, the “classical” methods of structural analysis, herein, are analytical methods rather than computational methods. We are still considering only elastic behavior.

Conjugate beam method

$$\upsilon' = \int \frac{M}{EI}dx = \text{slope} \quad (\theta = \int \frac{M}{EI}dx)$$

$$\upsilon = \int \upsilon' dx = \text{deflection} \quad (\delta = \int\int \frac{M}{EI}dxdx)$$

We create a “conjugate beam” and choose to load with $\overline{w} = \frac{M}{EI}$ where M is the moment along the actual beam (in terms of x).

The shear $\overline{V}$ in the conjugate beam $\overline{V} = -\int \overline{w}dx = -\int \frac{M}{EI}dx = \theta$ in the actual beam.

The moment $\overline{M}$ in the conjugate beam $\overline{M} = \int \overline{V}dx = -\int\int \frac{M}{EI}dxdx = \delta$ in the actual beam.

(positive M means load $\overline{w}$ acts downward)
(signs of δ and θ are usually obvious from inspection)

1. The slope at a given section of the actual beam equals the shear in the corresponding section of the conjugate beam.
2. The deflection at a given section of the actual beam equals the bending moment in the corresponding section of the conjugate beam.

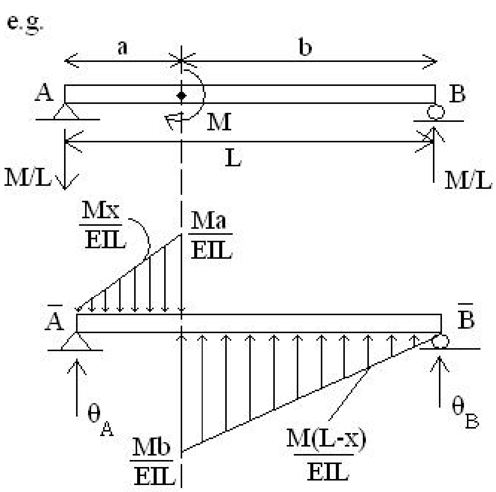
fixed end $\leftrightarrow$ free end
simple end $\leftrightarrow$ simple end (roller $\leftrightarrow$ roller)
internal connection $\leftrightarrow$ external interior support

i.e. if the actual beam has a fixed support, then it cannot rotate or deflect, so the conjugate beam is not allowed to have shear or moment $\therefore$ the conjugate beam needs a free end ($\theta = 0 \leftrightarrow \overline{V} = 0 \quad \delta = 0 \leftrightarrow \overline{M} = 0$)

The conjugate beam method is very fast for finding δ and θ at endpoints or supports, because δ and θ for the endpoints of the actual beam are just the support reactions of the conjugate beam.

note: It's okay if the conjugate beam appears unstable.

e.g.



$$\circlearrowleft^{+} \sum M_A : -M + R_B L = 0 \Rightarrow R_B = \frac{M}{L} = R_A$$

$$x = 0^+ : V(0^+) = -\frac{M}{L}$$

$$0^+ \le x \le a^- : M(x) = -\frac{Mx}{L}$$

$$x = a^- : M(a^-) = -\frac{Ma}{L}$$

$$x = a^+ : M(a^+) = -\frac{Ma}{L} + M = \frac{Mb}{L}$$

$$a^+ \le x \le L : \frac{Mb}{L} + \int_a^x -\frac{M}{L}dx = \frac{M(L-x)}{L}$$

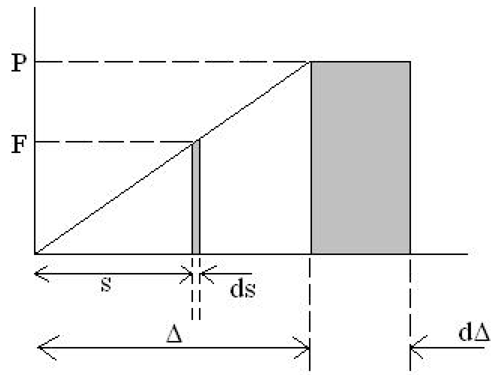
$$\textit{from } \sum M_{\bar{A}} = 0, \bar{V}_{\bar{B}} = \theta_B = \left[-\frac{Ma^2}{2L}\left(\frac{2}{3}a\right) + \frac{Mb^2}{2L}\left(a + \frac{b}{3}\right)\right]\frac{1}{EIL}$$

$$\textit{from } \sum M_{\bar{B}} = 0, \bar{V}_{\bar{A}} = \theta_A = \left[\frac{Mb^2}{2L}\left(\frac{2}{3}b\right) - \frac{Ma^2}{2L}\left(b + \frac{a}{3}\right)\right]\frac{1}{EIL}$$

External work and internal work

Consider a load gradually applied to a structure. Assume a linear relationship exists between the load and the deflection. This is the same assumption used in Hooke's Law in the previous chapter, and shown by experiment to be true within the "linear elastic" range for most materials.

Then, $W = \int_0^{\Delta} F ds = \frac{1}{2}P\Delta \quad \Delta =$ deflection (results in a triangular P versus Δ graph) note: If another force besides P occurs at the location of P, further $d\Delta$ will occur <u>without</u> further increasing the magnitude of P. P remains constant, so the additional work <u>done by</u> P is $P\,d\Delta$ (rectangular P versus Δ graph). This is important in the derivation of the unit load method later on.



Also, W = $\frac{1}{2}C\theta$ where C = external couple moment

This external work is converted to internal energy (strain energy)

dW = $\frac{1}{2}Md\theta$

Using $\frac{d\theta}{ds} = \frac{M}{EI}$ or $d\theta = \frac{Mds}{EI} \Rightarrow dW = \frac{M^2ds}{2EI}$

So, the total strain energy in the beam of length L, is

$$W = \int_0^L \frac{M^2dx}{2EI}$$

For a truss (axial force S only),

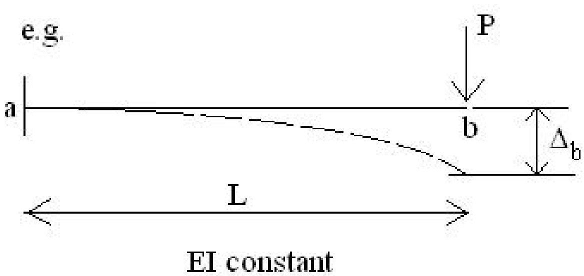
Strain energy W = $\frac{1}{2}S(dL) = \frac{1}{2}S(\frac{SL}{AE}) = \frac{S^2L}{2AE}$ per member.

So, $W = \sum \frac{S^2L}{2AE}$ for the entire truss.

<u>Equating External and Internal Work</u>

This concept can be used to find δ or θ at a point.

e.g.



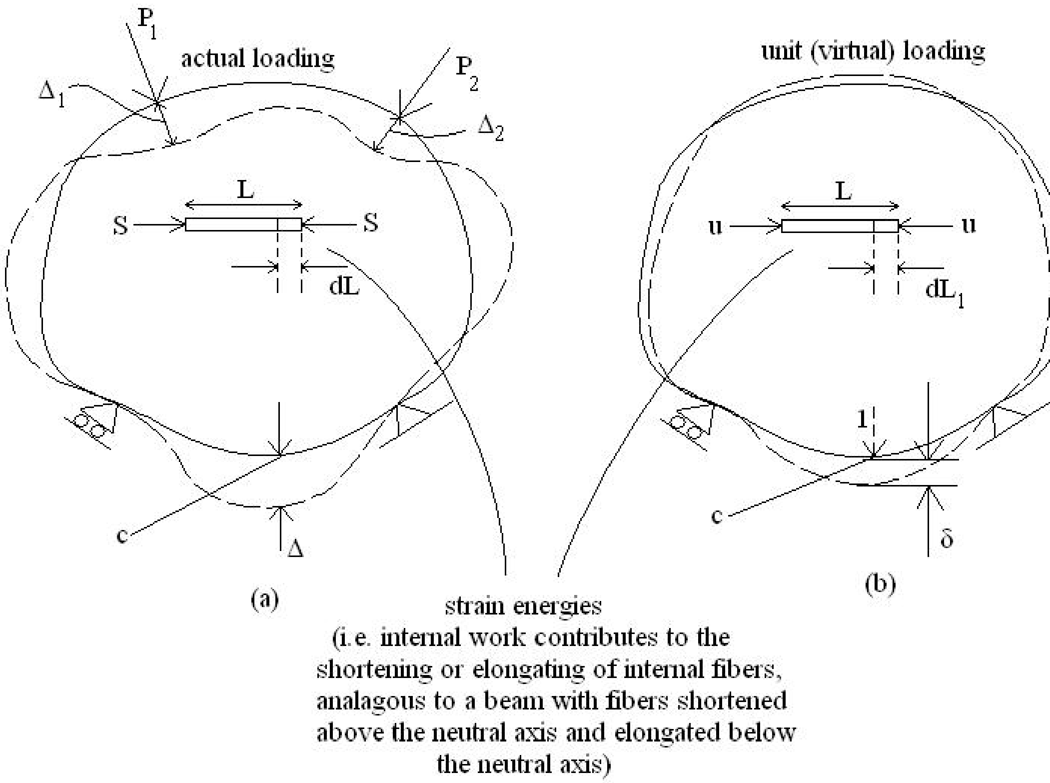
$$W_{external} = \frac{1}{2}Pa_b$$

$$W_{int\,ernal} = \int_0^L \frac{M^2dx}{2EI} = \int_0^L \frac{(-Px)^2dx}{2EI} = \frac{P^2L^3}{6EI}$$

$$Equating \Rightarrow a_b = \frac{\boldsymbol{PL^3}}{\boldsymbol{3EI}}$$

This method is quite limited in application since it is applicable only to deflection at a point of concentrated force. Also, if more than one force is applied to the system, a solution becomes impossible since there will be many deformations.

Method of virtual force (unit load method)



External work must equal the internal strain energy.

$$\frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 = \frac{1}{2}\sum S * dL$$

$$\frac{1}{2}(1)\,\delta = \frac{1}{2}\sum u * dL_1$$

Compared with the previous section, this is a more useful derivation of internal strain, which applies to multiple loads, none of which are required to be at the location in which we want to find the deflection.

Now imagine that the actual loads P_1 and P_2 are gradually applied to case “b”.

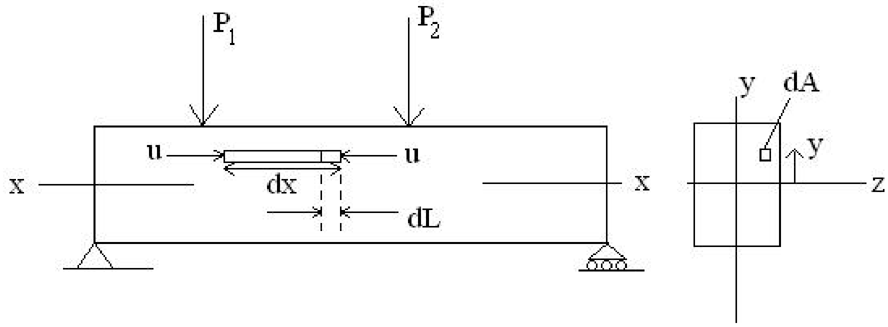
Equating external work and internal strain energy yields ;

$$\frac{1}{2}(1)\,\delta + \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + 1 * \Delta = \frac{1}{2}\sum u * dL_1 + \frac{1}{2}\sum S * dL + \sum u * dL$$

$1*\Delta$ and $\sum u*dL$ are the extra "rectangular" work values as described in the previous section.

The strain energy and work done must be the same whether the loads are applied together or separately, from conservation of energy.
$1*\Delta$ must cancel with $\sum u*dL$
i.e. $1*\Delta = \sum u*dL$ or $1*\theta = \sum u*dL$ where "1" in the second expression corresponds to an external unit couple.

note: "1"and "u" are virtual values and "Δ", "dL", and "θ" are actual values.



We need to find dL and u in terms of actual, measurable, quantities.
$\frac{My}{I} =$ stress at y
(stress) = (strain) E = $\frac{dL}{dx}(E)$ where dx = length of fiber
$\Rightarrow dL = \frac{(stress)(dx)}{E} = \frac{Mydx}{EI}$ and u = force = (stress)(area) = $\frac{my}{I}dA$
note: the upper case "M" corresponds to the moment from the "actual" values (moment resulting from P_1 and P_2 in the picture above), while the lower case "m" corresponds to the moment from the "virtual" unit force.

$$1*\Delta = \sum(\frac{my}{I}dA)(\frac{My}{EI}dx) = \int_0^L \frac{Mm}{EI^2}dx\int_A y^2 dA \quad \text{But, } \int_A y^2 dA = I$$

So,

$$\mathbf{1*\Delta = \int_0^L \frac{Mm}{EI}dx}$$

where m = bending moment from unit load and M = bending moment from actual loads

Also,

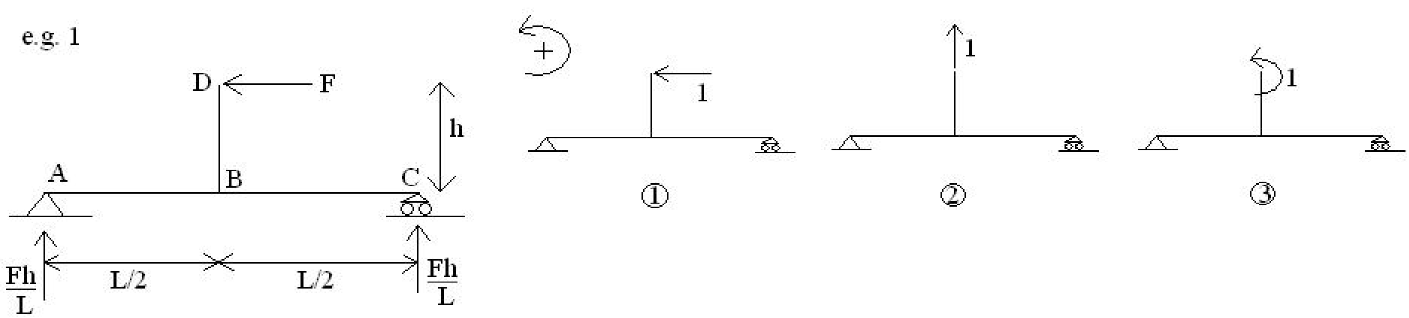
$$1*^{-} Z \int_0^L \frac{\mathbf{Mm}}{\mathbf{EI}} \mathbf{dx}$$

m = bending moment from unit *couple* and M = bending moment from actual loads

Now is a good time to recap some of the minor assumptions that may not have been explicitly stated so far:

Small angle approximations. These were used in the derivation bending stress formula, stress $= \frac{My}{I}$, which has been used in this section. Small angle approximations are valid for most structural engineering applications.
Neglecting of axial deformations. Nowhere in this section did we include axial stress and strain of the beam, only axial stress and strain of the "internal fibers." This will be shown in a later section to be a valid assumption.
Conservative forces. Consider a beam loaded by gravity. The beam will deform, and the forces will thus hit the beam at an angle. This curvature is ignored in our force analysis, since the difference in solutions is negligible, as long as the small angle approximation is valid. This assumption has been used in previous chapters, and will continue to be used in all later chapters as well.
Engineering strain, rather than true strain. This assumption was stated in the section on "Hooke's Law" in the previous chapter. This assumption was used in the current section since the strain of the "internal fibers" was taken to be $\frac{dL}{dx}$ rather than $\frac{dL}{(dx - dL)}$.
Engineering strain will continue to be used in later chapters, since the difference in solutions is almost always negligible.

e.g. 1



member	origin	M	m_1	m_2	m_3
BD	B	Fh - Fx	h - x	0	1
AB	A	$\frac{-Fh}{L}x$	$\frac{-h}{L}x$	$\frac{1}{2}x$	$-\frac{x}{L}$
BC	C	$\frac{Fh}{L}x$	$\frac{h}{L}x$	$\frac{1}{2}x$	$\frac{x}{L}$

$$deformation\ to\ left = \int_0^h \frac{(Fh - Fx)(h-x)}{EI}dx + \int_0^{L/2}\frac{(\frac{-Fh}{L})x(\frac{-hx}{L})}{EI}dx + \int_0^{L/2}\frac{(\frac{Fhx}{L})(\frac{hx}{L})}{EI}dx$$

$$= \frac{\mathbf{1}}{\mathbf{EI}}\boldsymbol{F(.333h^3}\ \text{H}\ \boldsymbol{.08333h^2 L)}\ \ (units\ of\ length)$$

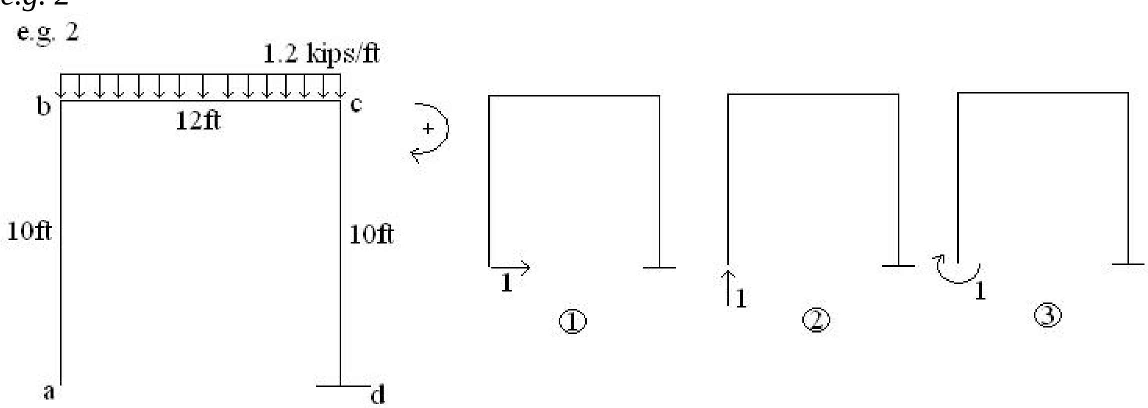
$$deformation\ upwards = 0 + \int_0^{L/2}\frac{(\frac{-Fh}{L})x(\frac{1}{2}x)}{EI}dx + \int_0^{L/2}\frac{(\frac{Fhx}{L})(\frac{1}{2}x)}{EI}dx = 0$$

$$rotation\ ccw = \int_0^h \frac{(Fh - Fx)(1)}{EI}dx + \int_0^{L/2}\frac{(\frac{-Fhx}{L})(\frac{-x}{L})}{EI}dx + \int_0^{L/2}\frac{(\frac{Fhx}{L})(\frac{x}{L})}{EI}dx$$

$$= \frac{\mathbf{1}}{\mathbf{EI}}\boldsymbol{F(.5h^2}\ \text{H}\ \boldsymbol{.08333hL)}\ \ (radians)$$

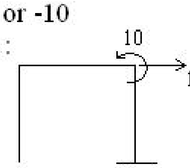
note: signs can be tricky, which is one of the reasons why this method should be limited to beams and very simple frames.
note: It doesn't matter where we take our "origins" as long as we're consistent.
note: This problem could have been solved using $\frac{d^2 3}{dx^2} = \frac{M}{EI}$ *with* $-_{BC} = -_{BA}$ *as a continuity condition.*

e.g. 2



member	origin	limit	M	m_1	m_2	m_3
ab	a	0 to 10	0	-x	0	1
bc	b	0 to 12	$\frac{-1.2x^2}{2}$	-10	x	1
cd	c	0 to 10	$\frac{-1.2(12)^2}{2}$ = -86.4	x - 10	12	1

note: m_1 here not simply equal to 0 or -10 because the reactions at c look like :



$$a_1 = \int \frac{Mm_1}{EI}dx = \frac{1}{EI}\left[0 + \int_0^{12}\left(\frac{-1.2x^2}{2}\right)(-10)dx + \int_0^{10}(-86.4)(x-10)dx\right]$$

$$= \mathbf{7776}\ \frac{kip*ft^3}{EI}\ \text{(right) (horizontal deflection at a)}$$

$$a_2 = \int \frac{Mm_2}{EI}dx = \frac{1}{EI}\left[0 + \int_0^{12}\left(\frac{-1.2x^2}{2}\right)(x)dx + \int_0^{10}(-86.4)(12)dx\right]$$

$$= \mathbf{-13478}\ \frac{kip*ft^3}{EI}\ \text{(down) (vertical deflection at a)}$$

$$a_3 = \int \frac{Mm_3}{EI}dx = \frac{1}{EI}\left[0 + \int_0^{12}\left(\frac{-1.2x^2}{2}\right)(1)dx + \int_0^{10}(-86.4)(1)dx\right.$$

$$= \mathbf{-1210}\ \frac{kip*ft^2}{EI}\text{(ccw)}$$

note: In this problem, as a whole, moments that deform the structure clockwise, are treated as positive. Those that deform ccw are negative. This determines the sign convention for M and m values. This is not to be confused with the sign convention for the solutions. Positive solutions $\Rightarrow$ deformation occurs in the directions assumed on the unit force diagrams.

Instead of “dL” being a non-measurable quantity associated with internal “fibers”, it can be the actual change in length of a truss member.

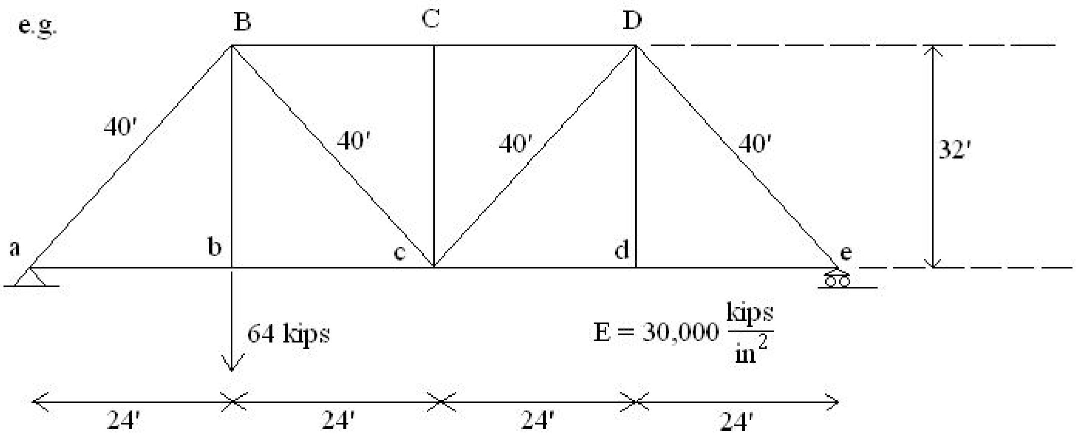
$1*\Delta = \sum u*dL \quad dL = \frac{SL}{AE}$ S = internal force in a given member due to actual loads

$$a = \sum_{1}^{m} \frac{\mathbf{SuL}}{\mathbf{AE}} \text{ (truss)}$$

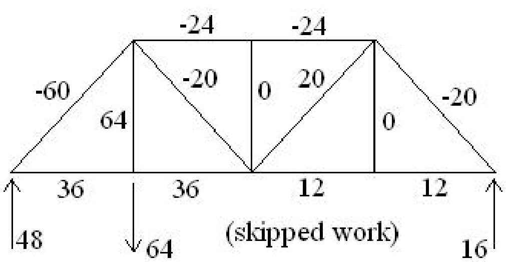
u = internal force in a given member due to a fictitious unit load at the point and in the direction where the deflection is sought

L = length of member
A = cross-sectional area of member
E = modulus of elasticity of member
m = total number of members

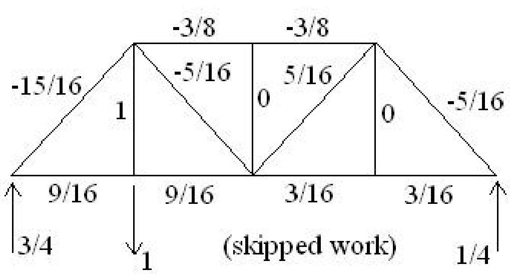
e.g.



Not all L or all A are the same, but for simplification the ratio L/A is always the same in this example.



forces from actual loading
(kips)



force from virtual unit loading
(kips)

member	S	u	Su
ab	36	9/16	20.25
bc	36	9/16	20.25
cd	12	3/16	2.25
de	12	3/16	2.25
BC	-24	-3/8	9
CD	-24	-3/8	9
aB	-60	-15/16	56.25
Bb	64	1	64
Bc	-20	-5/16	6.25
Cc	0	0	0
cD	20	5/16	6.25
Dd	0	0	0
De	-20	-5/16	6.25
			$\sum$ 202.0

$$a = \sum \frac{SuL}{AE} = \sum \frac{Su}{E}(1)$$

E is constant, so

$$\Delta = \frac{202}{30{,}000} = \mathbf{.00673\ ft\ (down)}$$

note: Finding the rotation of, for instance, member bc is equivalent to finding the relative displacement between ends b and c divided by the length bc.

Castigliano's second theorem

"the first partial derivative of the total strain energy of the structure with respect to one of the applied actions gives the displacement along that action"

$\Delta_P =$ corresponding displacement (deflection or rotation) along P,
where P = particular force or couple

$$\Delta_P = \frac{dW}{dP}$$

For a loaded beam, total strain energy W = $\int_0^L \frac{M^2 dx}{2EI}$ (derived in the section titled "External work and internal work")

For a loaded truss, total strain energy W = $\sum \frac{S^2 L}{2AE}$

Note: $M = M_1 + M_2 = m_1 P_1 + m_2 P_2$, where M = bending moment at any section
M_1 = moment at any section due to load P_1
m_1 = bending moment at any section due to a unit load in place of P_1

The fact that M_1 can be represented by the product of m_1 and P_1 (and the same for M_2, m_2, and P_2) and the fact that M can be represented by the sum of M_1 and M_2, are both important principles made possible by the principle of superposition.

Using the Chain Rule for derivatives (skipped work) ;

Castigliano's Theorem for beams : Castigliano's Theorem for trusses :

$$\Delta_1 = \int_0^L \frac{M\left(\frac{dM}{dP_1}\right)}{EI} dx \qquad \Delta_1 = \sum \frac{S\left(\frac{dS}{dP_1}\right)L}{AE}$$

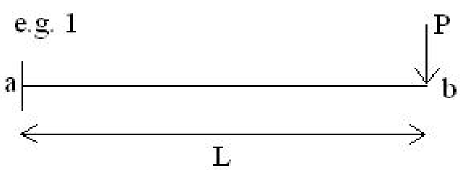
We can easily show that Castigliano's Theorem and the unit load method are really one in the same:

$$\Delta_1 = \frac{dW}{dP_1} = \frac{d}{dP_1}\int_0^L \frac{(m_1P_1 + m_2P_2)^2}{2EI}dx = \int_0^L \frac{2(m_1P_1 + m_2P_2)(m_1 + 0)}{2EI}dx = \int_0^L \frac{Mm_1}{EI}dx$$

Similarly, $\Delta_2 = \int_0^L \frac{Mm_2}{EI}dx$, as expected.

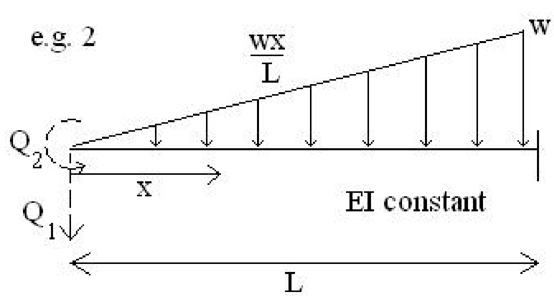
note: Theorem applies for θ calculations as well.

e.g. 1



$$a_P = a_b = \int_0^L \frac{(-Px)(-x)}{EI}dx = \boldsymbol{\frac{PL^3}{3EI}}$$

e.g. 2
Castigliano's Theorem clearly works with multiple loads and can also work at points where a load is not present, by placing an imaginary load P_1 at the point of interest and setting up the equation for a_1 in terms of P_1. Then, set $P_1 = 0$ either before or after integrating/summing.



$w = 1.5\, kN/m$

$L = 5m$

$E = 20000\,\frac{kN}{cm^2}$

$I = 12000\,cm^4$

$M = -Q_1x - Q_2 - \frac{wx^3}{6L}$

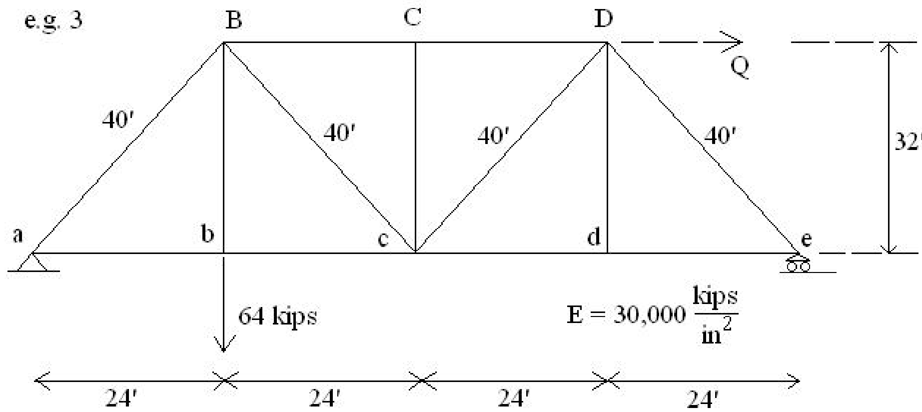
$\frac{dM}{dQ_1} = -x$ *and* $\frac{dM}{dQ_2} = -1$

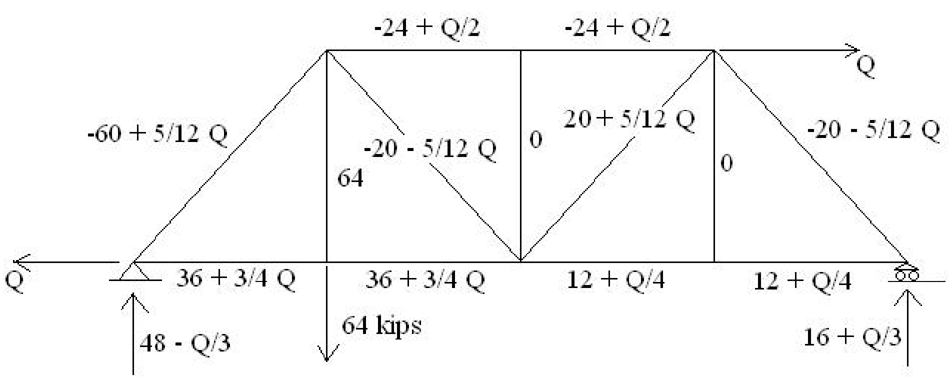
$$a_a = \int_0^L \frac{M(\frac{dM}{dQ_1})}{EI}dx = \int_0^L \frac{(-\frac{wx^3}{6L})(-x)}{EI}dx = \frac{wL^4}{30EI} = \textbf{1.3 cm (down)}$$

we're using $M = -\frac{wx^3}{6L}$ *since* $Q_1 = Q_2 = 0$

$$-_{a} = \int_0^L \frac{(-\frac{wx^3}{6L})(-1)}{EI}\,dx = \frac{wL^3}{24EI} = \boldsymbol{.003\ rad\ (ccw)}$$

e.g. 3   (same truss as the truss example in previous section)





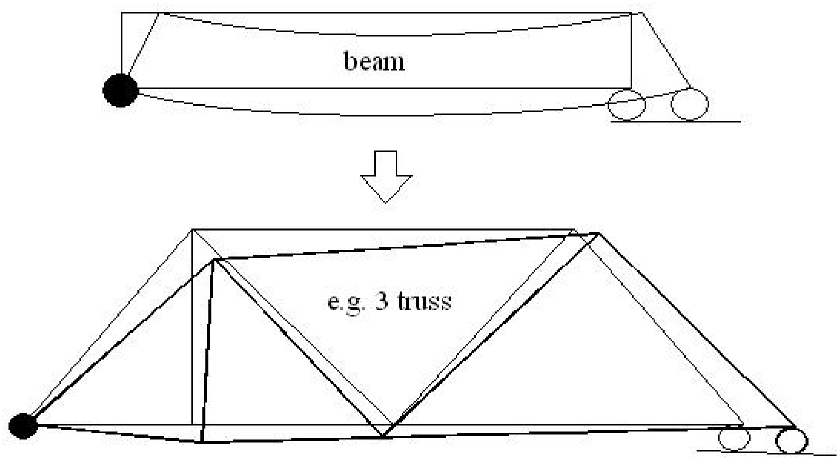
member	S	dS/dQ	$\frac{S(dS/dQ)L}{A}$	
ab	36 + 3/4 Q	3/4	27	(setting Q = 0)
bc	36 + 3/4 Q	3/4	27	
cd	12 + 1/4 Q	1/4	3	
de	12 + 1/4 Q	1/4	3	
BC	-24 + 1/2 Q	1/2	-12	
CD	-24 + 1/2 Q	1/2	-12	
aB	-60 + 5/12 Q	5/12	-25	
Bb	64	0	0	
Bc	-20 - 5/12 Q	-5/12	8.333	
Cc	0	0	0	
cD	20 + 5/12 Q	5/12	8.333	
Dd	0	0	0	
De	-20 - 5/12 Q	-5/12	8.333	
		$\sum$	36	

$$a = \sum \frac{S(dS/dQ)L}{AE} = \frac{36}{30000}$$

$= $ ***.0012 ft (right)***

note: This result may seem odd. Since the top chord is in compression, one might expect to see movement at joint D to the left.

The fact that joint D moves to the right is entirely due to the fact that our left pin is immovable and our right roller slides to the right. This is, in fact, exactly how bridges are constructed. Bridge structures are exposed to the elements, so the bridge sits on a roller to allow for expanding/contracting due to temperature fluctuations. (See the diagrams below) So, joint D <u>would</u> move to the right, as we've found.

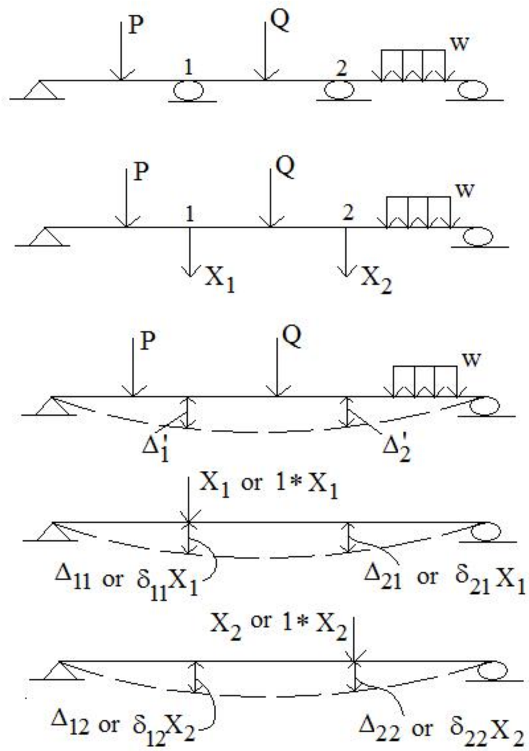


For beams in buildings, we should remember that the typical left pin/right roller is not the reality. Typical beams in buildings do not actually sit on rollers, but are pin/pin. The left pin/right roller idealization for typical beams yields exactly the same results as a pin/pin, however, because of all of our beam assumptions. These assumptions include assuming the supports are at the neutral axis (a perfectly valid assumption for typical beams) which PREVENTS any horizontal reaction from creating a moment due to eccentricity, the idea of conservative force which PREVENTS any horizontal reaction from creating a moment due to the beam's curvature, and the neglecting of axial deformation. All of these assumptions result in pin/pin and pin/roller yielding identical results. I.e. we can use the methods in the following sections for finding redundant forces and we would see that the horizontal force is zero for the case of a pin/pin beam with vertical loads, as would obviously be the case for a pin/roller. We can then use formulas we already know from this and previous sections for finding deflections and we would see that the vertical deflections are also identical for pin/pin versus pin/roller. Does this seem reasonable?

A beam that is pin-pin is restrained against horizontal motion, whereas a beam that is idealized as pin/roller or roller/roller is not restrained. Intuitively, this makes a difference. Intuitively, if subjected only to vertical force, the beam that is pin/pin will still have horizontal reactions whereas the beam that is roller/roller or pin/roller will not have horizontal reactions, since it is free to move. However, these horizontal reactions are small and, intuitively, are small enough to be neglected. In practice, the typical beams analyzed in this manner (simple supports) are checked for vertical force and vertical deflections, which intuitively would NOT be SIGNIFICANTLY affected by the pin/pin restraints. In practice, the typical beams are completely ignored when lateral drifts are checked. So, luckily, we don't need to conclude that all beam-related formulas up to this point are false and develop new formulas. The fact that pin/pin or pin/roller makes no difference for vertical forces, horizontal forces, and vertical deflections of our typical beams seems reasonable, because in reality the differences would in fact be very small.

Unlike typical beams, the supports of moment frames and braced frames and trusses should always be modeled exactly how they are actually built. As we will see in the following sections, when we consider a <u>*system*</u> *of beams or bars, such as a moment frame, horizontal reactions* <u>*will*</u> *be developed from vertical forces. We will also consider a pin/pin truss. Unlike a beam, a truss that is pin/pin* <u>*is*</u> *different than a truss that is pin/roller. A truss will have different forces and deflections, depending on the support configuration.*

Method of consistent deformations – Redundant forces



$\Delta'_1 + \Delta_{11} + \Delta_{12} = 0$

These *could* be combined into a single integral (so that M is in terms if P, Q, X_1, and X_2). Then, it would just be Δ_1 (from all forces) = 0.

Specifically, $\Delta_1 = \int_0^L \frac{M(\frac{dM}{dX_1})}{EI} dx$ (if using Castigliano's Theorem)

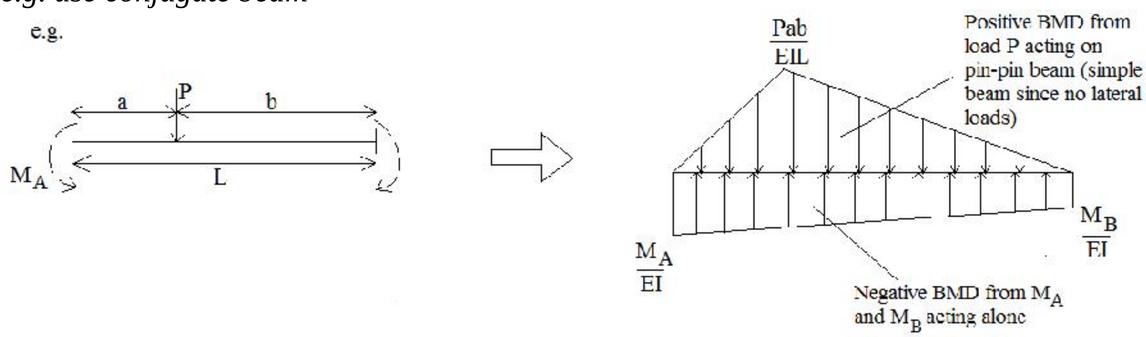
Also,

$\Delta'_2 + \Delta_{21} + \Delta_{22} = 0$
or
$\Delta'_1 + \delta_{11} X_1 + \delta_{12} X_2 = 0$ (1)
$\Delta'_2 + \delta_{21} X_1 + \delta_{22} X_2 = 0$ (2)

Δ = deflection due to external loads (with redundant supports removed).
δ_{11} = deflection at point 1 due to a unit force at point 1
δ_{12} = deflection at point 1 due to a unit force at point 2

This applies to couples and/or loads.
We can use Castigliano's Theorem, the unit load method, or any other method.
There are two unknowns X_1, X_2, and two equations (1), (2) $\Rightarrow$
Solve for the redundant supports. Then, find the rest of the support reactions.
(clearly X_1 and X_2 as pictured will have negative values)

e.g. use conjugate beam

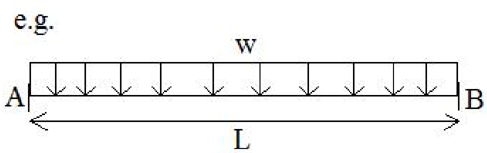


(conjugate beam)

From equilibrium of the loaded conjugate beam,

$$M_A = \frac{Pab^2}{L^2} \qquad ; \qquad M_B = \frac{Pa^2b}{L^2}$$

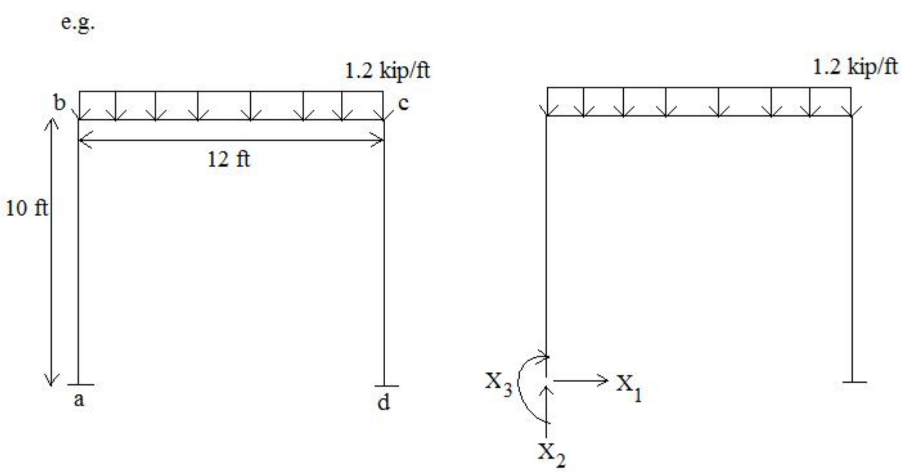
e.g. use superposition



We know that $M_A = \frac{Pab^2}{L^2}$ *for an arbitrary point load (derived in the previous example).*
For this uniform load, if P = wdx, a = x, and b = L-x,
then $M_A = \int_0^L \frac{w\,dx(x)(L-x)^2}{L^2} = \frac{1}{12}L^2 w$

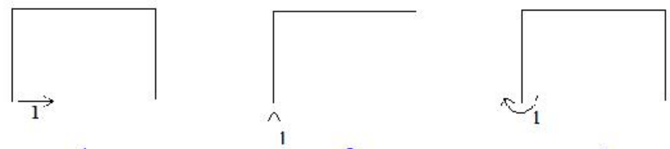
When faced with fixed-end beams or propped-cantilevered beams, reactions can be determined by this approach regardless of load distribution, as long as we know the reactions for an arbitrary point load (M_A *for a propped-cantilevered beam from an arbitrary point load =* $\frac{Pab(L+b)}{2L^2}$*)*

e.g. use unit load method



We can approach this problem by using a system of equations such as (1) and (2) on the previous page. We've already found " a_1 *", "* a_2 *", and "* a_3 *" from e.g. 2 in the "virtual force (unit load method)" section*

$$\begin{bmatrix}\Delta'_1\\ \Delta'_2\\ \Delta'_3\end{bmatrix} - \begin{bmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}$$



1 2 3

member	origin	m_1	m_2	m_3
ab	a	-x	0	1
bc	b	-10	x	1
cd	c	x-10	12	1

we need the deflections at "a" for the combined loading of the unit forces in 1, 2, 3

$$General:\ \text{¢} = \int_{ab}\frac{Mm}{EI}dx + \int_{bc}\frac{Mm}{EI}dx + \int_{cd}\frac{Mm}{EI}dx$$

¢_{11} : *deflection in direction 1 due to 1*

¢_{12} : *deflection in direction 1 due to 2*

¢_{13} : *deflection in direction 1 due to 3*

¢_{21} : *deflection in direction 2 due to 1*

¢_{22} : *deflection in direction 2 due to 2*

¢_{23} : *deflection in direction 2 due to 3*

¢_{31} : *deflection in direction 3 due to 1*

¢_{32} : *deflection in direction 3 due to 2*

¢_{33} : *deflection in direction 3 due to 3*

$(\text{¢}_a)_1 =$ *deflection at "a" in direction 1*

$$= \text{¢}_{11} + \text{¢}_{12} + \text{¢}_{13} = [\int_{ab}\frac{(m_1)^2}{EI}dx + \int_{bc}\frac{(m_1)^2}{EI}dx + \int_{cd}\frac{(m_1)^2}{EI}dx]$$

$$+ [\int_{ab}\frac{m_2m_1}{EI}dx + \int_{bc}\frac{m_2m_1}{EI}dx + \int_{cd}\frac{m_2m_1}{EI}dx] + [\int_{ab}\frac{m_3m_1}{EI}dx + \int_{bc}\frac{m_3m_1}{EI}dx + \int_{cd}\frac{m_3m_1}{EI}dx]$$

$(\text{¢}_a)_1 =$ *deflection at "a" in direction 2*

$$= \text{¢}_{21} + \text{¢}_{22} + \text{¢}_{23} = [\int_{ab}\frac{m_1m_2}{EI}dx + \int_{bc}\frac{m_1m_2}{EI}dx + \int_{cd}\frac{m_1m_2}{EI}dx]$$

$$+ [\int_{ab}\frac{(m_2)^2}{EI}dx + \int_{bc}\frac{(m_2)^2}{EI}dx + \int_{cd}\frac{(m_2)^2}{EI}dx] + [\int_{ab}\frac{m_3m_2}{EI}dx + \int_{bc}\frac{m_3m_2}{EI}dx + \int_{cd}\frac{m_3m_2}{EI}dx]$$

$(\text{¢}_a)_1 =$ *deflection at "a" in direction 3*

$$= \text{¢}_{31} + \text{¢}_{32} + \text{¢}_{33} = [\int_{ab}\frac{m_1m_3}{EI}dx + \int_{bc}\frac{m_1m_3}{EI}dx + \int_{cd}\frac{m_1m_3}{EI}dx]$$

$$+\left[\int_{ab}\frac{m_2 m_3}{EI}dx+\int_{bc}\frac{m_2 m_3}{EI}dx+\int_{cd}\frac{m_2 m_3}{EI}dx\right]+\left[\int_{ab}\frac{(m_3)^2}{EI}dx+\int_{bc}\frac{(m_3)^2}{EI}dx+\int_{cd}\frac{(m_3)^2}{EI}dx\right]$$

$$¢_{11}=\frac{1}{EI}\left[\int_0^{10}(-x)^2dx+\int_0^{12}(-10)(-10)dx+\int_0^{10}(x-10)dx\right]=1867\,{}^{kip*ft^3}\!/_{EI}\quad (right)$$

$$¢_{12}=\frac{1}{EI}\left[0+\int_0^{12}(-10)(x)dx+\int_0^{10}(x-10)(12)dx\right]=-1320\,{}^{kip*ft^3}\!/_{EI}\quad (left)$$

$$¢_{13}=\frac{1}{EI}\left[\int_0^{10}(-x)(1)dx+\int_0^{12}(-10)(1)dx+\int_0^{10}(x-10)(1)dx\right]=-220\,{}^{kip*ft^3}\!/_{EI}\quad (left)$$

$$¢_{21}=¢_{12}=-1320\,{}^{kip*ft^3}\!/_{EI}\quad (down)$$

$$¢_{22}=\frac{1}{EI}\left[0+\int_0^{12}x^2dx+\int_0^{10}12^2dx\right]=2016\,{}^{kip*ft^3}\!/_{EI}\quad (up)$$

$$¢_{23}=\frac{1}{EI}\left[0+\int_0^{12}(x)(1)dx+\int_0^{10}(12)(1)dx\right]=192\,{}^{kip*ft^3}\!/_{EI}\quad (up)$$

$$¢_{31}=¢_{13}=-220\,{}^{kip*ft^2}\!/_{EI}\quad (ccw)$$

$$¢_{32}=¢_{23}=192\,{}^{kip*ft^2}\!/_{EI}\quad (cw)$$

$$¢_{33}=\frac{1}{EI}\left[\int_0^{10}dx+\int_0^{12}dx+\int_0^{10}dx\right]=32\,{}^{kip*ft^2}\!/_{EI}\quad (cw)$$

$$\begin{bmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{bmatrix}=\begin{bmatrix}1867 & -1320 & -220\\ -1320 & 2016 & 192\\ -220 & 192 & 32\end{bmatrix}$$

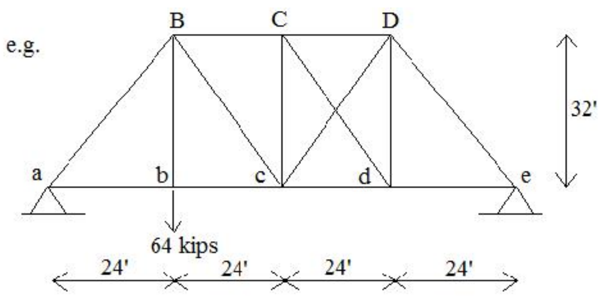
$$\begin{bmatrix}7776\\ -13478\\ -1210\end{bmatrix}+\begin{bmatrix}1867 & -1320 & -220\\ -1320 & 2016 & 192\\ -220 & 192 & 32\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\quad\Rightarrow\quad\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}=\begin{bmatrix}1.53\\ 7.20\\ 5.16\end{bmatrix}\begin{matrix}\text{kips}\\ \text{kips}\\ \text{kip*ft}\end{matrix}$$

note: $(1.2\frac{kips}{ft})(12ft) = 14.4\ kips$ *; From symmetry,* $\frac{14.4}{2} = 7.2kips$

note: Making a table greatly simplified this problem. Separating all of the deflections and summing is not necessary, but was done for clarity. The end result would be the same.

Using the method of consistent deformations in analyzing a frame would become intolerable if the problem involves as many redundant elements as a rigid frame usually does.

e.g.



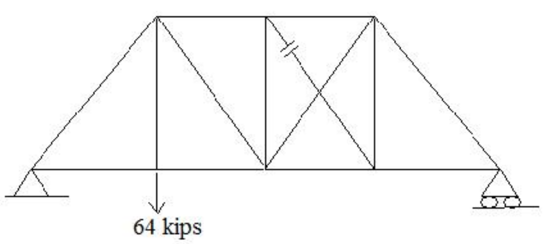
E = 30,000 $kips/in^2$

$\frac{L(ft)}{A(in^2)} = 1$ *for all members*

note: $14 + 4 > 2j = 16$

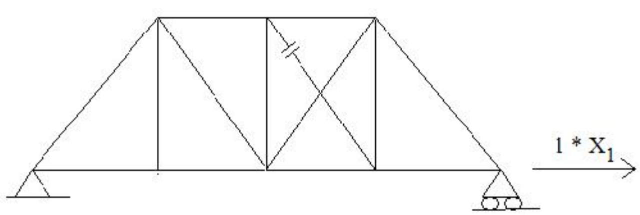
(redundant to the 2^{nd} *degree)*

Two redundant elements; one in the reaction component (choose "e") and the other in the bar (choose Cd).

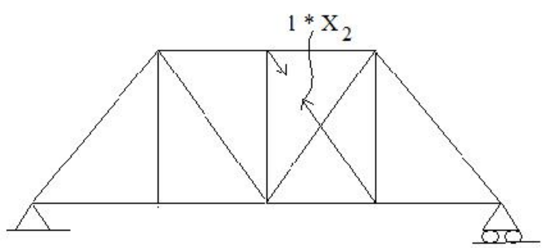


The horizontal movement at support e and the relative axial displacement between cut ends of bar Cd are zero.

One way to think of it is:
a$_2$ *and* ¢$_{21}$ *cause joints C and d to move* <u>*closer to each other*</u> *along the line Cd. The cut ends overlap.*



For beam Cd to be one piece, its unknown internal force X_2*, must shorten the beam by* ¢$_{22}$ *so that the cut ends no longer overlap. The end result is a shorter beam Cd, but no displacement between cuts.*



member	S'	u_1	u_2	$\frac{S'u_1L}{A}$	$\frac{S'u_2L}{A}$	$\frac{u_1^2L}{A}$	$\frac{u_2^2L}{A}$	$\frac{u_1u_2L}{A}$	$S = S' + u_1X_1 + u_2X_2$
ab	36	1	0	36	0	1	0	0	36 - 25.6 + 0 = 10.4
bc	36	1	0	36	0	1	0	0	36 - 25.6 + 0 = 10.4
cd	12	1	-3/5	12	-7.2	1	9/25	-3/5	12 - 25.6 + 6.4 = -7
de	12	1	0	12	0	1	0	0	12 - 25.6 + 0 = -13.6
BC	-24	0	0	0	0	0	0	0	-24 + 0 + 0 = -24
CD	-24	0	-3/5	0	14.4	0	9/25	0	-24 + 0 + 6.4 = -17.6
aB	-60	0	0	0	0	0	0	0	-60 + 0 + 0 = -60
Bb	64	0	0	0	0	0	0	0	64 + 0 + 0 = 64
Bc	-20	0	0	0	0	0	0	0	-20 + 0 + 0 = -20
Cc	0	0	-4/5	0	0	0	16/25	0	-20 + 0 + 8.5 = 8.5
Cd	0	0	1	0	0	0	1	0	0 + 0 - 10.6 = -10.6
cD	20	0	1	0	20	0	1	0	20 + 0 - 10.6 = 9.4
Dd	0	0	-4/5	0	0	0	16/25	0	0 + 0 + 8.5 = 8.5
De	-20	0	0	0	0	0	0	0	-20 + 0 + 0 = -20
			Σ	96	27.2	4	4	-3/5	
				Δ_1	Δ_2	δ_{11}	δ_{22}	$\delta_{12} = \delta_{21}$	

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Sigma \frac{S'u_1L}{AE} \\ \Sigma \frac{S'u_2L}{AE} \end{bmatrix} + \begin{bmatrix} \Sigma \frac{u_1^2L}{AE} & \Sigma \frac{u_2u_1L}{AE} \\ \Sigma \frac{u_1u_2L}{AE} & \Sigma \frac{u_2^2L}{AE} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 96 \\ 27.2 \end{bmatrix} + \begin{bmatrix} 4 & -3/5 \\ -3/5 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -25.6 \\ -10.6 \end{bmatrix} \text{kips}$$

member Cd is compressive

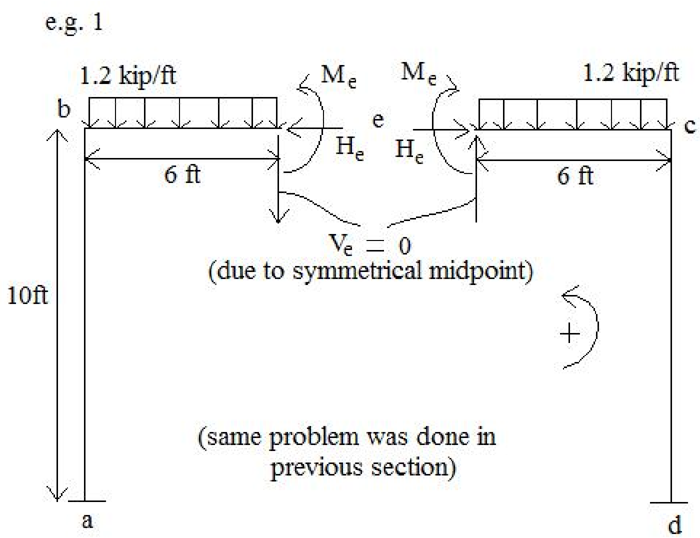
note: Deformation must always be considered when the truss is statically indeterminate. Using method of sections, for example, would not work because it would yield a singular solution.

Castigliano's compatibility equation (method of least work) – Redundant forces

As mentioned at the beginning of the previous section, another way of finding redundants is through Castigiliano's Theorem:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} = \begin{bmatrix} \frac{dW}{dX_1} \\ \frac{dW}{dX_2} \\ \cdot \\ \cdot \\ \frac{dW}{dX_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

e.g.1



Since e is a symmetrical midpoint,

$$\frac{dW}{dM_e} = 0 \text{ or } \int \frac{M(\frac{dM}{dM_e})}{EI} dx = 0$$

(slope at e = 0
)
Our other condition will be

$$\frac{dW}{dH_e} = 0 \text{ or } \int \frac{M(\frac{dM}{dH_e})}{EI} dx = 0$$

member	origin	limit	M	$\frac{dM}{dM_e}$	$\frac{dM}{dH_e}$
eb	e	0 to 6	$M_e - \frac{1.2x^2}{2}$	1	0
ba	b	0 to 10	$M_e - H_e x - 21.6$ (21.6 due to 1.2 kip/ft load)	1	x

"2" is in the numerator of the following expressions because we're only using half of the frame:

$$\frac{dW}{dM_e} = 0 \Rightarrow \frac{2}{EI}[\int_0^6 (M_e - \frac{1.2x^2}{2})(1)dx + \int_0^{10}(M_e + H_e x - 21.6)(1)dx] = 0$$

$$\Rightarrow .96M_e + 3H_e - 15.552 = 0 \qquad (1)$$

$$\frac{dW}{dH_e} = 0 \Rightarrow \frac{2}{EI}[\int_0^{10}(M_e + H_e x - 21.6)(x)dx] = 0$$

$$\Rightarrow 3M_e + 20H_e - 64.8 = 0 \qquad (2)$$

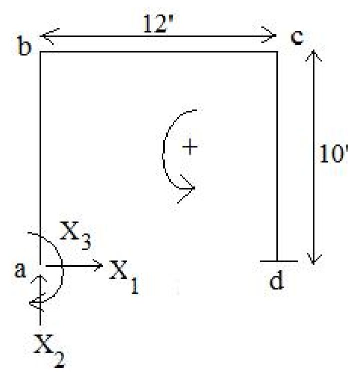
Solving (1) and (2) yields $H_e = 1.525 kips, M_e = 11.4353 kips * ft$

From equilibrium, $H_A = 1.53 kips; M_A = 5.08 kips$ *(skipped work)*
This is exact to two decimals. Calculations for this problem in the previous section were rounded, resulting in the slight difference in solution of M_A.

This example could just as easily of been solved by using the unit load method at the cut. Method of least work is still too cumbersome for a highly redundant frame.

e.g. 2 – Same a previous example, but apply the method of least work at "a" and

take $\Delta_a = \begin{bmatrix} 7776 \\ -13478 \\ -1210 \end{bmatrix}$ *to be known.*



$$M = M_{ab} + M_{bc} + M_{cd}$$
$$= [-X_3 + X_1 x] + [-X_3 - X_2 x + 10X_1] + [-X_3 - 12X_2 + (10X_1 - X_1 x)]$$

$$a_1 = \frac{1}{EI}[\int_0^{10}(-X_3 + X_1 x)(\frac{dM}{dX_1} = x)dx + \int_0^{12}(-X_3 - X_2 x + 10X_1)(10)dx$$
$$+ \int_0^{10}(-X_3 - 12X_2 + 10X_1 - X_1 x)(10 - x)dx] = 1867X_1 - 1320X_2 - 220X_3 \frac{kip * ft^3}{EI}$$

$$a_2 = \frac{1}{EI}[\int_0^{10}(-X_3 + X_1 x)(\frac{dM}{dX_2} = 0)dx + \int_0^{12}(-X_3 - X_2 x + 10X_1)(-x)dx$$
$$+ \int_0^{10}(-X_3 - 12X_2 + 10X_1 - X_1 x)(-12)dx] = -1320X_1 + 2016X_2 + 192X_3 \frac{kip * ft^3}{EI}$$

$$a_3 = \frac{1}{EI}[\int_0^{10}(-X_3 + X_1 x)(\frac{dM}{dX_3} = -1)dx + \int_0^{12}(-X_3 - X_2 x + 10X_1)(-1)dx$$

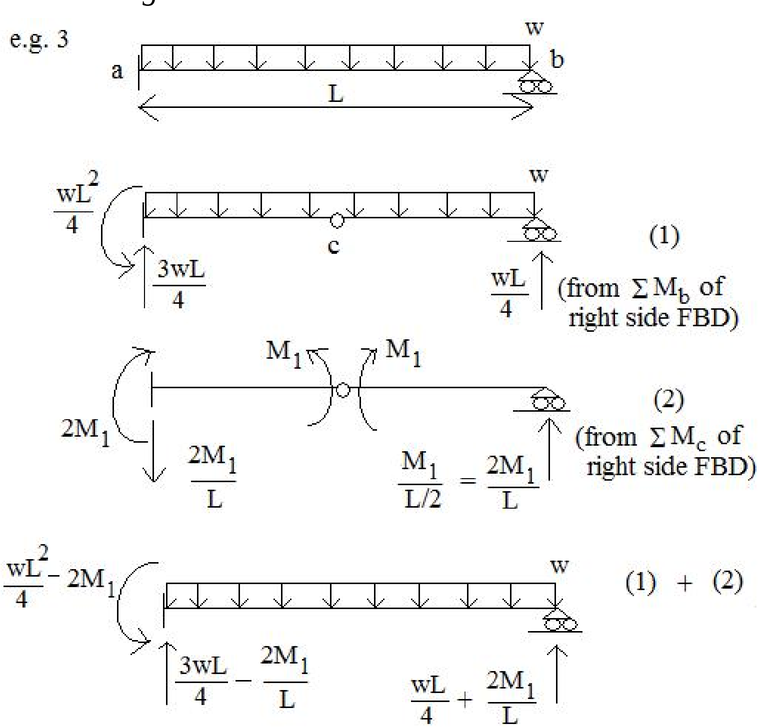
$$+\int_0^{10}(-X_3-12X_2+10X_1-X_1x)(-1)dx\,]=-220X_1+192X_2+32X_3\,\frac{kip*ft^2}{EI}$$

$$\begin{bmatrix}7776\\-13478\\-1210\end{bmatrix}+\begin{bmatrix}1867X_1-1320X_2-220X_3\\-1320X_1+2016X_2+192X_3\\-220X_1+192X_2+32X_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}=\begin{bmatrix}1.53\\7.20\\5.16\end{bmatrix}\begin{matrix}\text{kips}\\\text{kips}\\\text{kip*ft}\end{matrix}$$

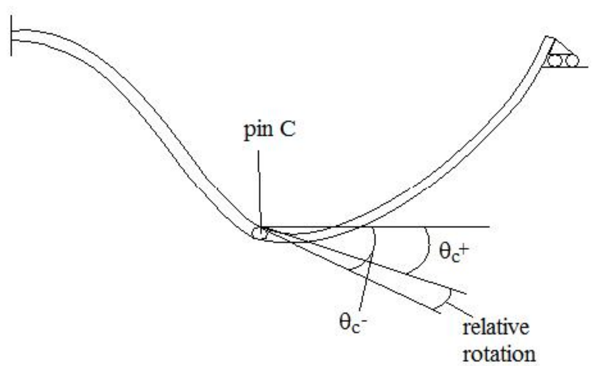
note: Δ_a *did not have to be worked out separately – the moments M = 0, M =* $\frac{1.2x^2}{2}$*, and M = 86.4 could have been added to each respective integrand. The end result would be the same.*

note: regardless of what deflections you choose to separate, or whether you use the unit load method or Castigliano's Theorem, separating the moments in a table, before integrating, is often a good idea.

e.g.3



reactions on the left in (1) and (2) are found from equilibrium of the whole beam after $\frac{wL}{4}$ *and* $\frac{2M_1}{L}$ *are known, respectively.*



Our redundant unknown is a <u>*ccw*</u> *moment couple just to the left of the pin, which decreases the relative rotation (makes* θ_c^- *less steep) and an equal* <u>*cw*</u> *moment couple just to the right of the pin, which decreases the relative rotation (makes* θ_c^+ *more steep).*

So, we call these moments M_1 *as shown in (2). In other words, we defined an* M_1 *that causes some* <u>*rotation*</u> $(\bar{\ }_c^{\,-} - \bar{\ }_c^{\,+}$ *to cancel the effect of the pin). In the real beam, this* <u>*rotation*</u> *is zero. So,* $\dfrac{dW}{dM_1} =$ <u>*rotation*</u> *from* $M_1 = 0$.

$x = 0^+ :$

$$V(0^+) = \frac{3wL}{4} - \frac{2M_1}{L} \qquad M(0^+) = \frac{-wL^2}{4} + 2M_1$$

$0^+ \le x \le L^- :$

$$V(x) = \frac{3wL}{4} - \frac{2M_1}{L} - \int_0^x w dx = \frac{3wL}{4} - \frac{2M_1}{L} - wx$$

$$M(x) = \frac{-wL^2}{4} + 2M_1 + \int_0^x \frac{3wL}{4} - \frac{2M_1}{L} - wx \, dx = \frac{-.5[Lw(x^2 - 1.5Lx + .5L^2) + 4M_1(x-L)]}{L}$$

(equivalent to finding the BMD for (1) and (2) separately and summing)

$x = L^- :$

$$V(L^-) = \frac{-wL}{4} - \frac{2M_1}{L} \qquad M(L^-) = 0$$

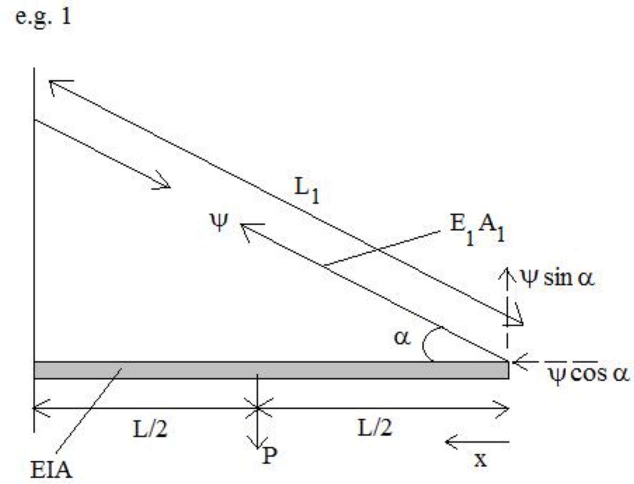
$$\frac{dW}{dM_1} = \frac{1}{EI}\int_0^L \left(\frac{-.5[Lw(x^2 - 1.5Lx + 4M_1(x-L)]}{L}\right)\left(\frac{-2(x-L)}{L}\right)dx = 0$$

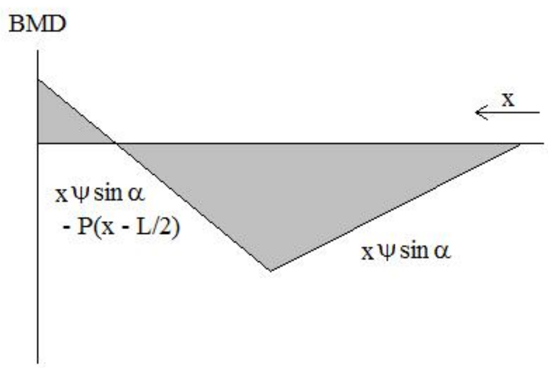
$$\Rightarrow M_1 = \frac{wL^2}{16} \quad \therefore \quad R_b = \frac{wL}{4} + \frac{2(\frac{wL^2}{16})}{L} = \frac{3}{8}Lw$$

note: Axial deformation is neglected for beams and frames in the method of consistent deformations, least work, and other methods to come. But what effect does this have on the accuracy of the calculations of redundant forces?

Composite Structure

e.g. 1





$$W = \int \frac{M^2 dL}{2EI} + \Sigma \frac{S^2 L_1}{2AE}$$

$$\frac{dW}{d\vee} = 0$$

$$W = \int_0^{L/2} \frac{x^2 \vee^2 \sin^2 \sim}{2EI} dx$$

$$+ \int_{L/2}^{L} \frac{(x\vee \sin\sim - P(x - L/2))^2}{2EI} dx$$

$$+ \frac{\vee^2 \cos^2 \sim L}{2AE} + \frac{\vee^2 L_1}{2A_1 E_1}$$

$$\frac{dW}{d\vee} = \int_0^{L/2} \frac{x^2 \vee \sin^2 \sim}{EI} dx$$

$$+ \int_{L/2}^{L} \frac{[x\vee \sin\sim - P(x - L/2)](x \sin\sim)}{EI} dx$$

$$+ \frac{\vee \cos^2 \sim L}{AE} + \frac{\vee L_1}{A_1 E_1} = 0$$

SOLVE

$$\Rightarrow \vee = \frac{P \sin\sim (\frac{5}{48}) L^3}{(\sin^2 \sim \frac{L^3}{3} + \cos^2 \sim \frac{IP}{A} + \frac{I L_1 E}{E_1 A_1})}$$

(axial deformation and bending)

note: If $\sim \rightarrow 90°$ *and* $E_1 \rightarrow \infty$, *then we have a prop-cantilevered beam with* $\vee = \frac{5P}{16}$

Choose values:

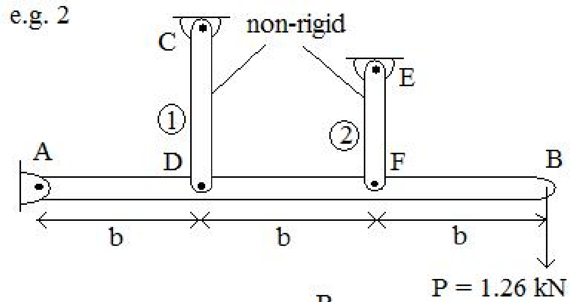
$E = E_1$, $P = 20$, $\sim = 30°$, $L = 10$, $L_1 = 12$, $I = \frac{®}{64}(.3)^4$, $A = ®(.3)^2$, $A_1 = ®(.1)^2$

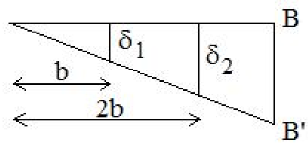
$\Rightarrow$ *(skip work)*

$\vee = 12.47$ *(with axial deformation)*

$\vee = 12.50$ *(without inclusion of axial deformation)*

e.g. 2





Axial Deformation Only :
DC and FE deform axially;
No bending allowed in ADFB

$E_1 = E_2$

$L_1 = .4m \quad d_1 = .004m$

$L_2 = .3m \quad d_2 = .003m$

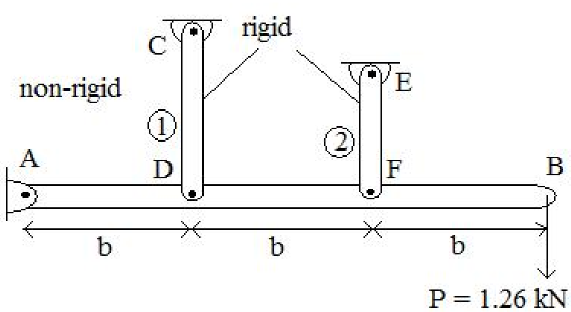
compatibility method:

$\overset{+}{\circlearrowleft} \sum M_A : T_1 b + T_2(2b) - 1.26(3b) = 0 \quad (1)$

$$\frac{\delta_2}{2b} = \frac{\delta_1}{b} \Rightarrow \delta_2 = 2\delta_1 \qquad \frac{T_2(.3)}{E(\frac{\pi}{4}(.003)^2)} = \frac{2T_1(.4)}{E(\frac{\pi}{4}(.004)^2)} \qquad (2$$

$T_2 = 1.4175kN \quad T_1 = .945kN$ *(axial deformation only)*
(different lengths and areas will result in different solutions)

Bending Only :

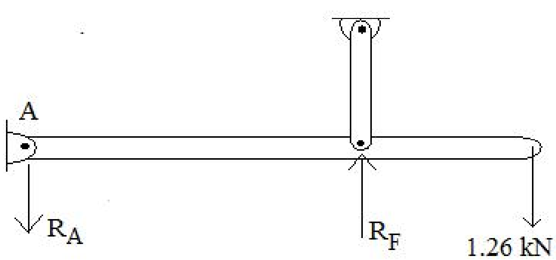


ADFB deforms in bending ; no axial deformation allowed in DC or FE

Unit load method :

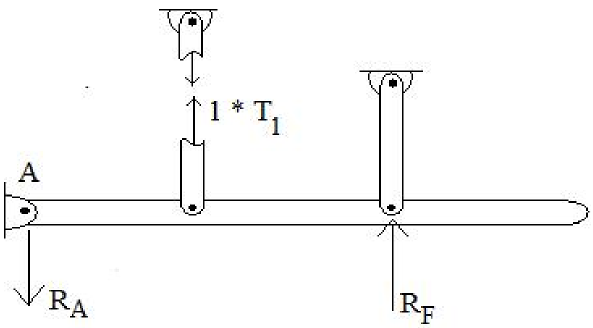
$\overset{+}{\circlearrowleft} \sum M_A : R_F(2b) - 1.26(3b) = 0 \Rightarrow R_F = 1.89kN$

$\uparrow^{+} \sum F_y : 1.89 - 1.26 - R_A = 0 \Rightarrow R_A = .63kN$



$0 \le x \le 2b^- : \quad M(x) = -.63x$

$2b^+ \le x \le 3b^- : \quad M(x) = 1.26x - 3.78b$



$\circlearrowleft^{+} \sum M_A : 1(b) - R_F(2b) = 0 \Rightarrow R_F = .5kN$

$\uparrow^{+} \sum F_y : 1 - 1/2 - R_A = 0 \Rightarrow R_A = .5kN$

$0 \le x \le b^- : M(x) = -.5x$

$b^+ \le x \le 2b^- : M(x) = .5x - 6$

$2b^+ \le x \le 3b : M(x) = 0$

$$a = \int_0^b \frac{(-.63x)(-.5x)}{EI}dx + \int_b^{2b}\frac{(-.63x)(.5x-b)}{EI}dx + \int_{2b}^{3b}\frac{(1.26x - 3.78b)(0)}{EI}dx = \frac{.315b^3}{EI}$$

$$¢ = \int_0^b \frac{(-.5x)^2}{EI}dx + \int_b^{2b}\frac{(.5x-b)^2}{EI}dx + 0 = \frac{.1667b^3}{EI}$$

$$a + ¢T_1 = 0 \qquad \frac{.315b^3}{EI} + \frac{.1667b^3}{EI}T_1 = 0$$

$T_1 = -1.890kN$ $\quad T_2 = 2.835kN$ (from equilibrium or from same process on T_2)
(bending only)

To treat the structure as a composite member, we need the moment of inertia I for beam ADFB, as well as "b" :

Take $I = \frac{®}{64}(.006)^4 m^4, b = .3m$

Choose T_1 as the redundant:

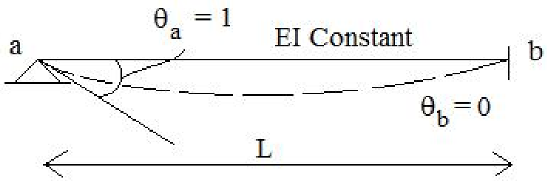
Total elongation of member DC = $\left(\frac{.315b^3}{EI} + \frac{.16667b^3T_1}{EI}\right) + \frac{2T_1(.4)}{E(\frac{®}{4}(.004)^2)} = 0$

$\Rightarrow T_1 = -1.888kN$

The inclusion of axial deformation has no <u>significant</u> effect on the solution, for this problem.

note: Previous analysis of the statically indeterminate one bay frame (with the beam loaded uniformly) resulted in each vertical support reaction being equal to exactly one-half of the total load (obviously the exact answer) despite the neglecting of axial deformation. This is because this frame was symmetrical (there are no relative axial displacements).

Moment distribution method – Joint moments in a frame



Take the propped-cantilevered beam shown. It has no load. We'd like to know the general relationship between θ_a and M_{ab} (end moment of member ab at "a") or M_{ba} (end moment of member ab at "b")

$$EI\frac{d^4\upsilon}{dx^4} = \omega = 0$$

Initial conditions: $\upsilon(0) = 0 \quad \upsilon(L) = 0 \quad \upsilon'(0) = \theta_a \quad \upsilon'(L) = 0$

$$\upsilon = x(1-\frac{x}{L})^2\theta_a \quad \text{(skipped work)}$$

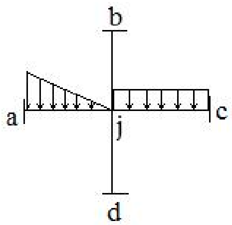
$$\frac{d^2\upsilon}{dx^2} = \frac{-M}{EI} \Rightarrow M(x) = E\frac{I}{L}(4-6\frac{x}{L})\theta_a$$

So, $M_{ab} = 4E\frac{I}{L}\theta_a$ and $M_{ba} = 2E\frac{I}{L}\theta_a$

OR

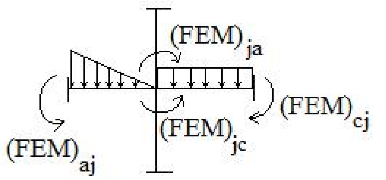
$M_{ab} = S_{ab}\theta_a$, where $S_{ab} = S_{ba}$ = member stiffness = $4Ek_{ab}$

where $k_{ab} = k_{ba}$ = stiffness factor $= \frac{I}{L}$



(a)

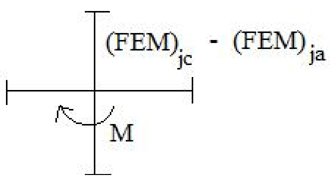
Lock the structure so that there are four fixed-end beams. Find the FEMs, including the total moment M at the center (ccw). Create an opposite moment shown (cw moment M) to "unlock" the beam. Joint j now rotates through an angle θ. Now, picture (e) is equivalent to picture (a) and we can proceed with the moment distribution.



$$m_{ja} = S_{ja}\theta$$
$$m_{jb} = S_{jb}\theta$$
$$m_{jc} = S_{jc}\theta$$
$$m_{jd} = S_{jd}\theta$$
(1)

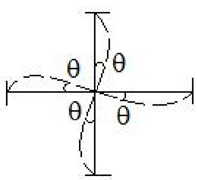
Sum of the moments must equal zero, so, $(S_{ja} + S_{jb} + S_{jc} + S_{jd})\,\theta = M$, where M is the "external" moment at joint j

(all θ are equal due to continuity)



If the members have the same E, but not necessarily the same I or L, then $4E(k_{ja} + k_{jb} + k_{jc} + k_{jd})\,\theta = M \Rightarrow \theta = \dfrac{M}{4E\sum k}$ (2)

$\sum k$ includes all members that connect at rotating joint (can vary depending on which end of beam)



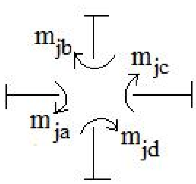
From (1) and (2),

$$m_{ja} = S_{ja}\frac{M}{4E\sum k} = \frac{4EI}{L}\frac{M}{4E\sum k} = \frac{k_{ja}}{\sum k}M = D_{ja}M$$

$$m_{jb} = \frac{k_{jb}}{\sum k}M = D_{jb}M$$

$$m_{jc} = \frac{k_{jc}}{\sum k}M = D_{jc}M$$

$$m_{jd} = \frac{k_{jd}}{\sum k}M = D_{jd}M$$



(e)

These are called distributed moments (DMs). D = distribution factor = $\dfrac{k}{\sum k}$

Assumes constant E – usually the case since beams made of different materials are rarely connected together.

note: D depends only on member dimensions. The individual moments are just ratios of each other that add up to M – i.e. the "external" moment, M, is *distributed* among the connecting beams, according to their relative dimensions (stiffnesses).

$m_{aj}, m_{bj}, m_{cj}, m_{dj}$ - called the carry-over-moments, need to be found.

M_{ab} and M_{ba}, which were found on the previous page, can be equated with a "carry-over-factor" ; $M_{ba} = C_{ab} M_{ab}$; $C_{ab} = C_{ba} = \frac{1}{2}$

So, $m_{aj} = \frac{1}{2} m_{ja}$; $m_{bj} = \frac{1}{2} m_{jb}$; $m_{cj} = \frac{1}{2} m_{jc}$; $m_{dj} = \frac{1}{2} m_{jd}$

These are called carry-over-moments (COMs).

$M_{ja} = (FEM)_{ja} + m_{ja}\ (= DM) + m_{ja}\ (= COM)$

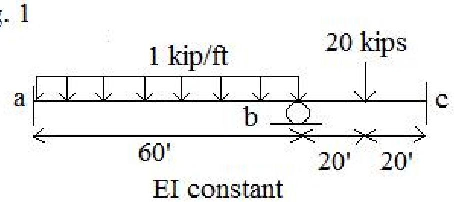
$M_{aj} = (FEM)_{aj} + m_{aj}\ (= DM) + m_{aj}\ (= COM)$

note: It is not yet clear how to find end moments for a frame when there is more than one joint that can rotate

note: sign conventions will become clear in the following examples

e.g. 1

e.g. 1



beam ba: $\dfrac{I/60}{I\,(1/60 + 1/40)} = .4$

beam bc: $\dfrac{I/40}{I\,(1/60 + 1/40)} = .6$

Clockwise moments = positive (FEMs can be found in Appendix B)

$(FEM)_{ba} = 300 \quad (FEM)_{ab} = -300 \quad (FEM)_{bc} = -100 \quad (FEM)_{cb} = 100$ *(skipped work)*

$M_b = -(300 + (-100)) = -200$ *(Total FEM at b)*

Distributed moments: $m_{ba} = .4(-200) = -80 kips * ft$; $m_{bc} = .6(-200) = -120 kips * ft$

Carry-over-moments: $m_{ab} = \frac{1}{2} m_{ba} = -40 kips * ft$; $m_{cb} = \frac{1}{2} m_{bc} = -60 kips * ft$

(COM has same sign as DM)

	ab	ba	bc	cb
$\frac{k}{\Sigma k}$	0	.4	.6	0
FEM	-300	300	-100	100
DM		-80	-120	
COM	-40			-60
Σ	-340	220	-220	40

$M_{ab} = -340 kips * ft$

$M_{ba} = 220 kips * ft$

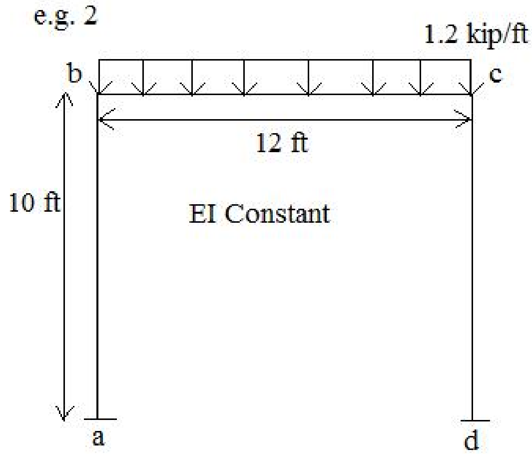
$M_{bc} = -220 kips * ft$

$M_{cb} = 40 kips * ft$

If there are multiple rotating joint, then the joints must be continually locked and unlocked until the carry-over-moments are considered negligible (see the following example)

note: For all cycles : At a fixed support, DM is zero. At a joint across from a fixed support, COM is zero. For a span with no load, FEM is zero (this does not necessarily mean that M = 0 for that span).

e.g. 2



$$Beam\ ba = \frac{I/10}{I/10 + I/12} = .545 = Beam\ cd$$

$$Beam\ bc = \frac{I/12}{I/10 + I/12} = .4545 = Beam\ cb$$

$$(FEM)_{bc} = \frac{-1.2(12)^2}{12} = -14.4 kips * ft$$

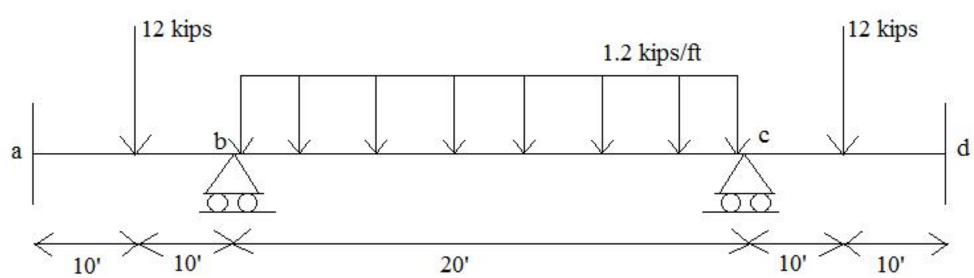
$$(FEM)_{cb} = 14.4 kips * ft$$

	(rigid) ab	(joint b) ba	bc	(joint c) cb	cd	(rigid) dc
$k/\Sigma k$	0	.545	.4545	.4545	.545	0
FEM			-14.4	14.4		
DM		14.4(.545) = 7.85	14.4(.4545) = 6.545	-6.545	-7.85	
COM	7.85/2 = 3.93		-6.545/2 = -3.27	3.27		-3.93
DM		3.27(.545) = 1.785	3.27(.4545) = 1.49	-1.49	-1.785	
COM	1.785/2 = .8926		-1.49/2 = -.744	.744		-.8926
DM		.744(.545) = .406	.744(.4545) = .338	-.388	-.406	
COM	.406/2 = .203		-.338/2 = -.169	.169		-.203
DM		.169(.545) = .092	.169(.4545) = .077	-.077	-.092	
COM	.092/2 = .046		-.077/2 = -.038	.038		-.046
DM		.038(.545) = .021	.038(.4545) = .017	-.017	-.021	
COM	.021/2 = .01		-.017/2 = -.009	.009		-.01
DM		.009(.545) = .005	.009(.4545) = .004	-.004	-.005	
Σ	5.08	10.2	-10.2	10.2	-10.2	-5.08

(after 6 cycles) (kip * ft)

e.g. 3

e.g. 3



	ab	ba	bc	cb	cd	dc
$\frac{k}{\Sigma k}$	0	.5	.5	.5	.5	0
FEM	-30	30	-40	40	-30	30
DM		5	5			
COM	2.5			2.5		
DM				-6.25	-6.25	
COM			-3.13			-3.13
DM		1.57	1.57			
COM	.78			.78		
DM				-.39	-.39	
COM			-.20			-.20
DM		.10	.10			
COM	.05			.05		
DM				-.03	-.03	
Σ	-26.67	36.67	-36.67	36.67	-36.67	26.67

(kips * ft)

Since cb and cd have not yet been "balanced" by a distributed moment, the C.O.M. is added to M_c (10 + 2.5) = 12.5

From this point on, the C.O.M.'s become the sole FEMs, because all rotational joints have been balanced by distributed moments

OR

(release joints "simultaneously")

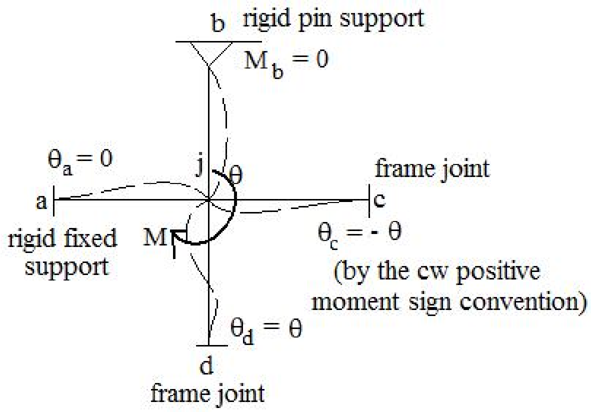
FEM	-30	30	-40	40	-30	30
DM		5	5	-5	-5	
COM	2.5		-2.5	2.5		-2.5
DM		1.25	1.25	-1.25	-1.25	
COM	.63		-.63	.63		-.63
DM		.32	.32	-.32	-.32	
COM	.16		-.16	.16		-.16
DM		.08	.08	-.08	-.08	
COM	.04		-.04	.04		-.04
DM		.02	.02	-.02	-.02	
Σ	-26.67	36.67	-36.67	36.67	-36.67	26.67

(kips * ft)

Joints are balanced. From this point on, the C.O.M.'s are the only FEMs.

note: So far, joint translations are ignored in the moment distribution method. This can have an effect on the accuracy of joint moments. Previous analysis of the one bay frame (with beam uniformly loaded) resulted in the exact solution because it is symmetrical (and hence there are no relative displacements in the columns), and there is no side sway, from inspection. Lateral loading and/or non-symmetrical gravity loading can cause joint translations.

Modified stiffness method – Shortcut for certain special cases



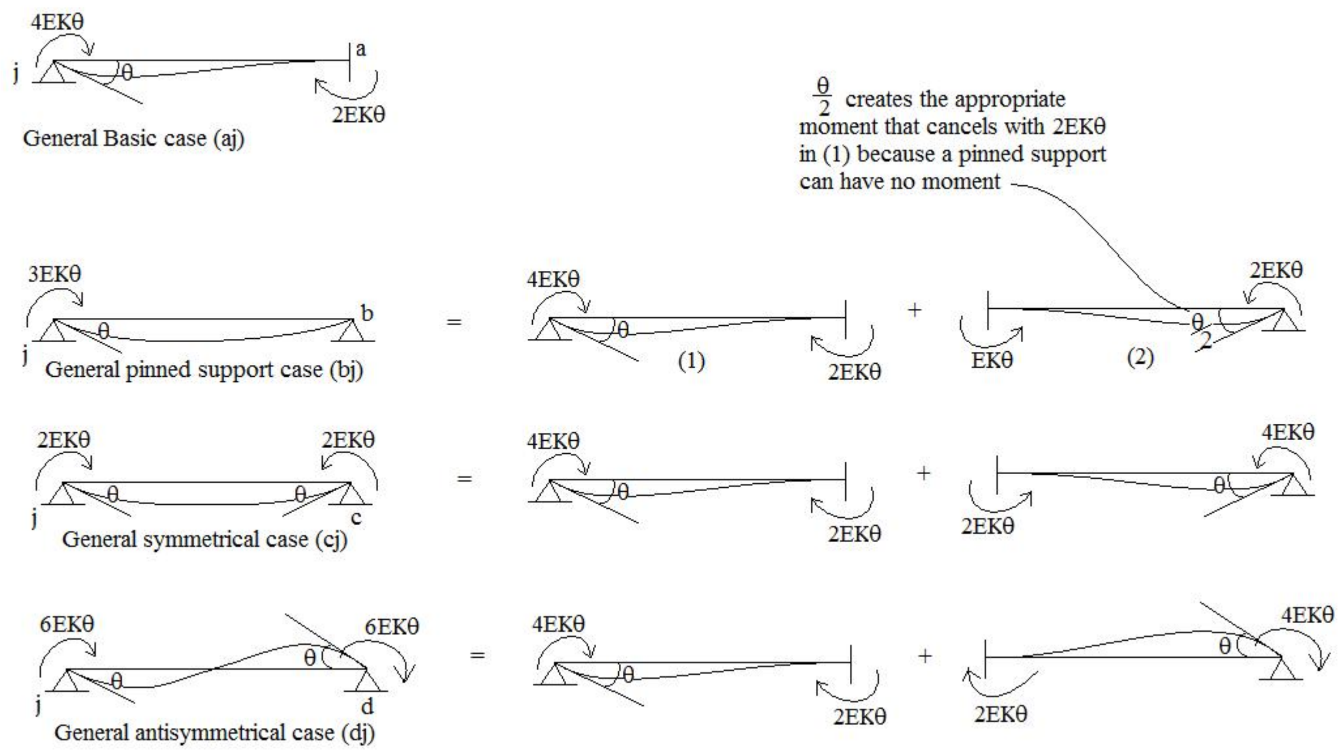
Special cases :

aj = the most basic member with a single DM at ja and COM at aj

bj = supported by a pin

cj = symmetrical

dj = antisymmetrical



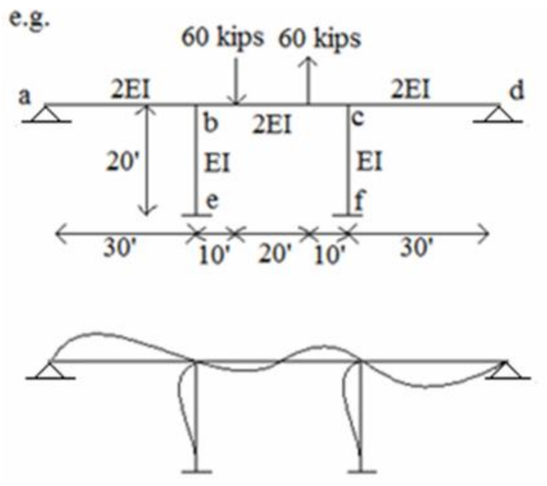
Basic: $k_{ja}' = k_{ja}$, where k' = "modified stiffness factor"

Pinned support : $M_{jb} = 3Ek_{jb}\theta = 4Ek_{jb}'\theta$, where $k_{jb}' = \frac{3}{4}k_{jb}$

Symmetrical : $M_{jc} = 2Ek_{jc}\theta = 4Ek_{jc}'\theta$, where $k_{jc}' = \frac{1}{2}k_{jc}$

Antisymmetrical : $M_{jd} = 6Ek_{jd}\theta = 4Ek_{jd}'\theta$, where $k_{jd}' = \frac{3}{2}k_{jd}$

$$M_{ja} = \frac{k_{ja}'}{\sum k'}M \qquad M_{jb} = \frac{k_{jb}'}{\sum k'}M \qquad M_{jc} = \frac{k_{jc}'}{\sum k'}M \qquad M_{jd} = \frac{k_{jd}'}{\sum k'}M$$



$$FEM_{bc} = FEM_{cb} = (\frac{10}{40})^2(\frac{30}{40})(60)(40)$$
$$-(\frac{10}{40})(\frac{30}{40})^2(60)(40) = -225kips$$
$$k_{ba} = \frac{2I}{30} = \frac{I}{15} \quad k_{bc} = \frac{2I}{40} = \frac{I}{20} = k_{cb}$$
$$k_{cd} = \frac{2I}{30} = \frac{I}{15} \quad k_{be} = \frac{I}{20} \quad k_{cf} = \frac{I}{20}$$

modified stiffness:

$k_{ba}' = \frac{3}{4}k_{ba} = .05I = k_{cd}'$ *(pinned/roller support)*

$k_{bc}' = \frac{3}{2}k_{bc} = .075I = k_{cb}'$ *(antisymmetrical)*

$k_{be}' = k_{be} = .05I = k_{cf}'$

	ab	eb	be	ba	bc	cb	cd	cf	fc	dc
$\frac{k'}{\Sigma k}$	1	0	.2857	.2857	.4286	.4286	.2857	.2857	0	1
FEM					-225	-225				
DM			64.3	64.3	96.4	96.4	64.3	64.3		
COM		32.15							32.15	
Σ	0	32.15	64.3	64.3	-128.6	-128.6	64.3	64.3	32.15	0

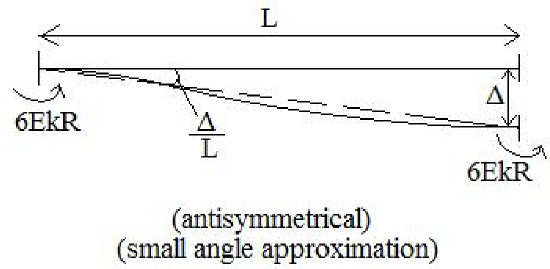
(kips * ft)

note: only one cycle needed for this problem with modified stiffness approach

note: since the frame is symmetrical, we don't really need to tabularize all of the moments, but rather just half of the frame

This problem is solved without modified stiffness in Hsieh (1995).

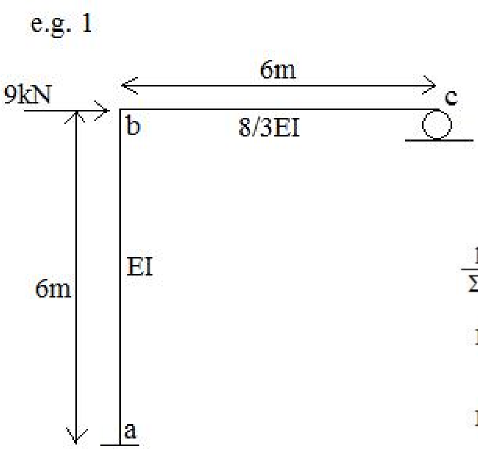
Treatment of joint translations



If there are joint translations, then
$M_{ja} = DM_{ja} + COM_{ja} + FEM_{ja}$
+ additional joint translation moment

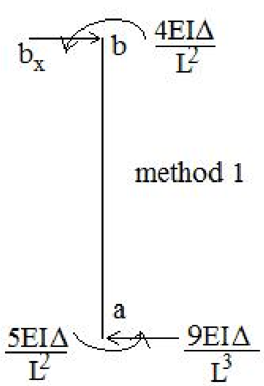
relative displacement, shown, results in negative moment, shown
$R = \Delta / L$

e.g. 1



$$\frac{(8/3)I}{6}\left(\frac{3}{4}\right) = k_{bc}' \quad \frac{I}{6} = k_{ba}'$$

	ab	ba	bc	cb
$\frac{k'}{\Sigma k'}$	0	1/3	2/3	1
FEM	$\frac{-6EI\Delta}{L^2}$	$\frac{-6EI\Delta}{L^2}$		
DM		$\frac{2EI\Delta}{L^2}$	$\frac{4EI\Delta}{L^2}$	
COM	$\frac{EI\Delta}{L^2}$			
Σ	$\frac{-5EI\Delta}{L^2}$	$\frac{-4EI\Delta}{L^2}$	$\frac{4EI\Delta}{L^2}$	0



method 1 $\quad \frac{9EIa}{L^3}$ *from* $\Sigma M_b = 0$ *for member ab*

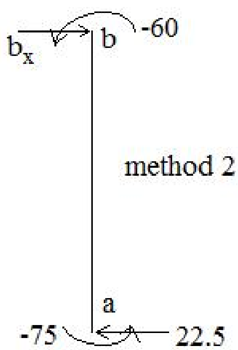
From $\Sigma F_x = 0$ *for the whole frame,* $\frac{9EIa}{L^3} = 9 \Rightarrow a = \frac{L^3}{EI}$

$$\Rightarrow M_{ab} = -30kN*m \quad M_{ba} = -24kN*m \quad M_{bc} = 24kN*m$$

OR

*Choose arbitrary FEM = -90 kN*m*

	ab	ba	bc	cb
FEM	-90	-90		
DM		30	60	
COM	15			
Σ	-75	-60	60	0



22.5 from $\sum M_b = 0$ *for member ab*

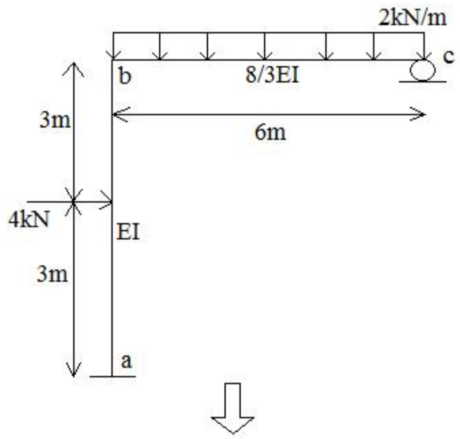
From $\sum F_x = 0$ *for the whole frame, 22.5* $\neq$ *9*

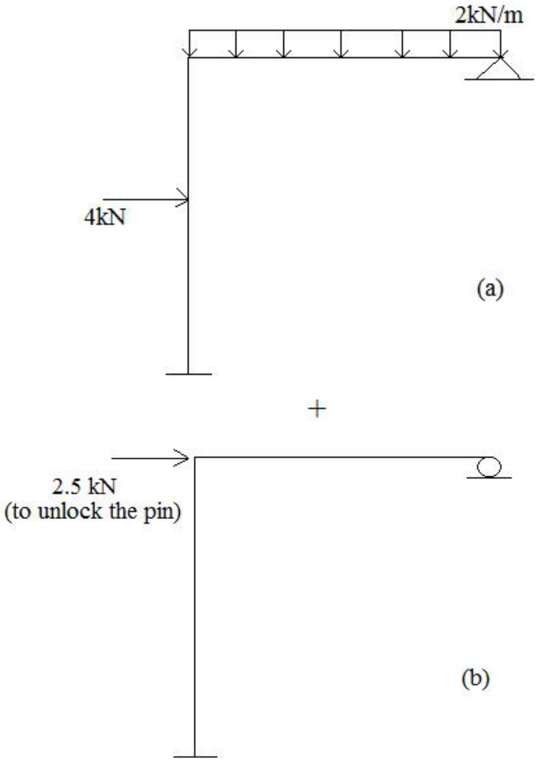
Need correction factor of $9/22.5$ *for all forces and moments*

$\Rightarrow \quad M_{ab} = -30kN*m \quad M_{ba} = -24kN*m \quad M_{bc} = 24kN*m$

note: method 2 does not even use the formula $M = -6EkR$*. Since the solutions for method 1 and 2 match correctly, we know our derivation for* $M = -6EkR$ *is correct. Method 2 is more manageable, but how can we use method 2 if the loads are* <u>*not*</u> *only located at the joints?*

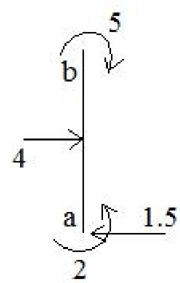
e.g. 2





(a)

	ab	ba	bc	cb
$\frac{k'}{\Sigma k'}$	0	1/3	2/3	1
FEM	-3	3	-6	6
DM		1	2	-6
COM	.5		-3	
DM		1	2	
COM	.5			
Σ	-2	5	-5	0

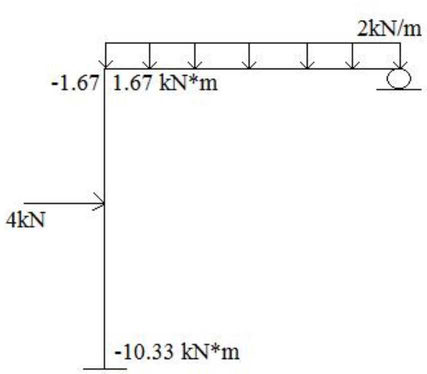


From $\sum F_x = 0$ for the whole frame, the pin has a reaction of 2.5kN acting left. This must be counter-acted as shown in (b) since the beam does <u>not</u> have a pin.

(from $\sum M_b$ for member ab)

(b)

	ab	ba	bc	cb
FEM	-90	-90		
DM		30	60	
COM	15			
Σ	-75	-60	60	0



From $\sum M_b = 0$ for member ab and $\sum F_x = 0$ for the whole frame, the correction factor is $^{2.5}/_{22.5}$, where 22.5 is from the previous example.

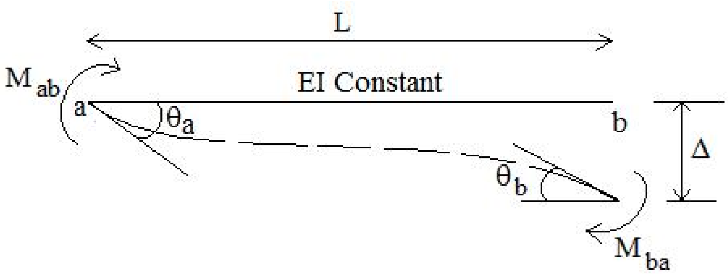
So, moments for b:

ab	ba	bc	cb
–8.33	–6.67	6.67	0

(a) + (b) $\Rightarrow$ $M_{ab} = -10.33$ $\quad M_{ba} = -1.67$ $\quad M_{bc} = 1.67\ kN * m$

note: moment distribution can be used to find moments that include the effect of sway for asymmetrical vertical loadings, and for multi-story frames, but it quickly becomes cumbersome to do by hand.

Slope-deflection method – Joint moments in a frame



Consider member ab, which is isolated from a loaded rigid frame (not shown).

Sign convention: all values shown are positive.

$$M_{ab} = \frac{4EI\theta_a}{L} + \frac{2EI\theta_b}{L} - \frac{6EI\Delta}{L^2} + (FEM)_{ab}$$
$$M_{ba} = \frac{2EI\theta_a}{L} + \frac{4EI\theta_b}{L} - \frac{6EI\Delta}{L^2} + (FEM)_{ba}$$

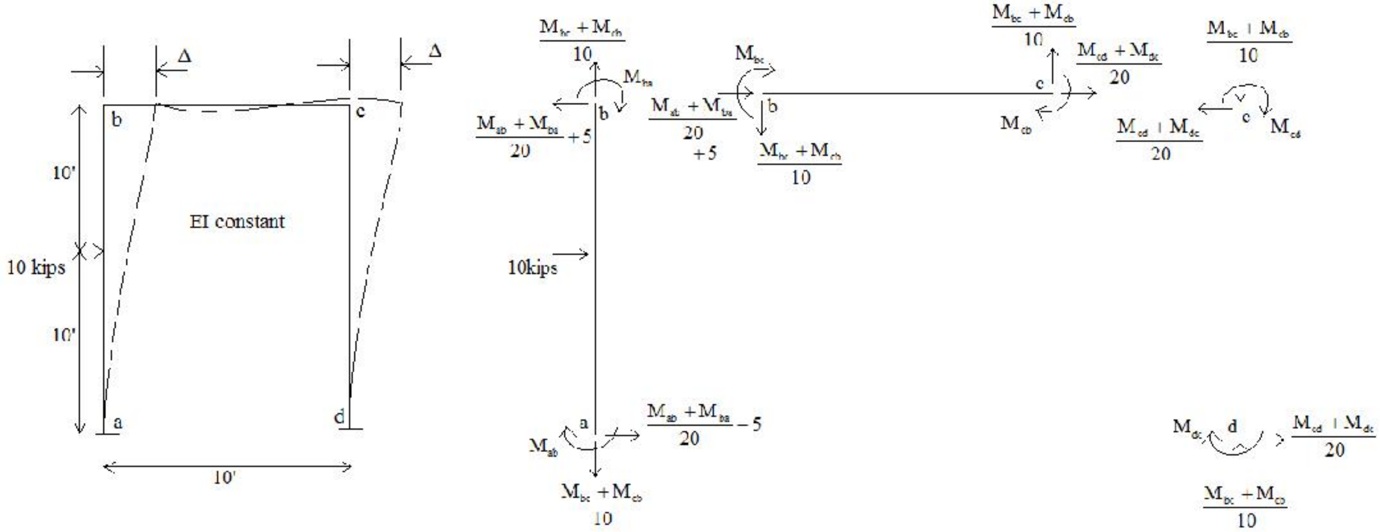
Recall these values from the derivations in the sections on “Moment distribution method” and “Treatment of joint translations”.

Manipulating a bit;

$$M_{ab} = 2Ek_{ab}(2\theta_a + \theta_b - 3R_{ab}) + FEM_{ab}$$
$$M_{ba} = 2Ek_{ba}(2\theta_b + \theta_a - 3R_{ba}) + FEM_{ba}$$
$$R = \frac{\text{Relative displacement}}{L} = \frac{\Delta}{L} \ (R_{ab} = R_{ba})$$

If joint “a” is a rigid support, then $\theta_a = 0$ (i.e. $(DM)_{ab} = 0$)

e.g.1



Assume end moments are known. This is sufficient to find all forces at the joints, from equilibrium. The result (all forces in terms of end moments) is shown. But now there is only one <u>*equilibrium*</u> *equation left (proof in e.g. 3) and we don't yet know the moments.*

$$M_{ab} = 2Ek_{ab}(^{-}_{b} - 3R) - FEM_{ab} = 2E(\frac{I}{20})(^{-}_{b} - 3R) - 25$$

$$M_{ba} = 2E(\frac{I}{20})(2^{-}_{b} - 3R) + 25$$

For M_{ab} *and* M_{ba}, $^{-}_{a} = 0$ *since "a" is a rigid support*

$$M_{bc} = 2E(\frac{I}{10})(2^{-}_{b} + ^{-}_{c})$$

$$M_{cb} = 2E(\frac{I}{10})(2^{-}_{c} + ^{-}_{b})$$

For M_{cb} *and* M_{bc}, *R = 0 and FEM = 0 (no relative displacement since axial deformation in ab and cd is neglected) (beam bc is also unloaded)*

$$M_{cd} = 2E(\frac{I}{20})(2^{-}_{c} - 3R)$$

$$M_{dc} = 2E(\frac{I}{20})(^{-}_{c} - 3R)$$

For M_{cd} *and* M_{dc}, $^{-}_{d} = 0$ *and FEM = 0 (rigid support at "d")(beam cd is unloaded)*

All R are equal for this frame (axial deformation in bc is neglected).

Including $^{-}_{b}, ^{-}_{c}, R(or\ \frac{a}{20})$, *there are 9 unknowns.*

Equations:

$$6\ above + (M_{ba} = -M_{bc}) + (M_{cb} = -M_{cd}) + [10 + (\frac{M_{ab} + M_{ba}}{20} - 5) + (\frac{M_{cd} + M_{dc}}{20}) = 0]$$

=9 equations

The last term in the above expression comes from $\sum F_x = 0$ for the whole structure, which is the extra equation of equilibrium mentioned previously.

Solving, yields:

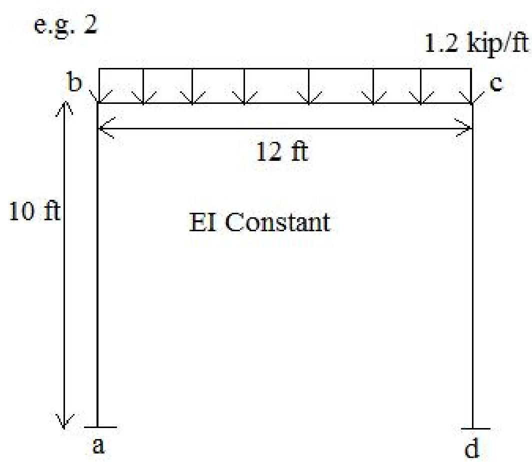
$$-_{c} = \frac{50.48}{EI} \quad -_{b} = \frac{-12.02}{EI} \quad R = \frac{92.95}{EI}$$

$$M_{ab} = -54.10 \quad M_{ba} = -5.30 \quad M_{bc} = 5.30 \quad M_{cb} = 17.80 \quad M_{cd} = -17.80 \quad M_{dc} = -22.80$$

*(kip * ft) (positive = cw)*

All joint forces have already been found in terms of the moments, so they can be solved for numerically, and bending moment diagrams can be drawn.

e.g. 2



$$M_{ab} = 2E\frac{I}{10}(^{-}{}_{b})$$

$$M_{ba} = 2E\frac{I}{10}(2^{-}{}_{b})$$

For member ab, $^{-}{}_{a} = 0$ *, R = 0, FEM = 0*

$$M_{bc} = 2E\frac{I}{12}(2^{-}{}_{b} + ^{-}{}_{c}) - 14.4$$

$$M_{cb} = 2E\frac{I}{12}(2^{-}{}_{c} + ^{-}{}_{b}) + 14.4$$

For member bc, R = 0

$$M_{cd} = 2E\frac{I}{10}(2^{-}{}_{c})$$

$$M_{dc} = 2E\frac{I}{10}(^{-}{}_{c})$$

For member cd, $^{-}{}_{d} = 0$ *, R = 0, FEM = 0*

Additional equations: $M_{ba} = -M_{bc} \quad M_{cd} = -M_{cb}$

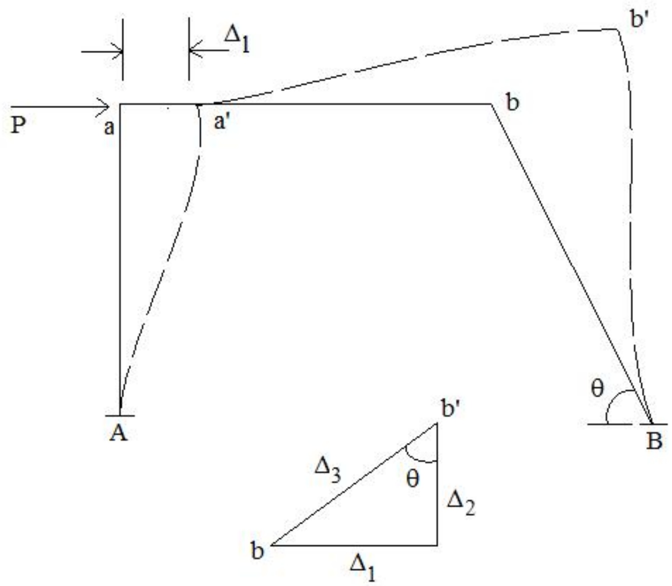
Solve $\Rightarrow \; -_{b} = \frac{25.4}{EI} \quad -_{c} = \frac{-25.4}{EI}$

$$M_{ab} = 5.08 \quad M_{ba} = 10.16 \quad M_{bc} = -10.16 \quad M_{cb} = 10.16 \quad M_{cd} = -10.16 \quad M_{dc} = -5.08$$

*(kip * ft)*

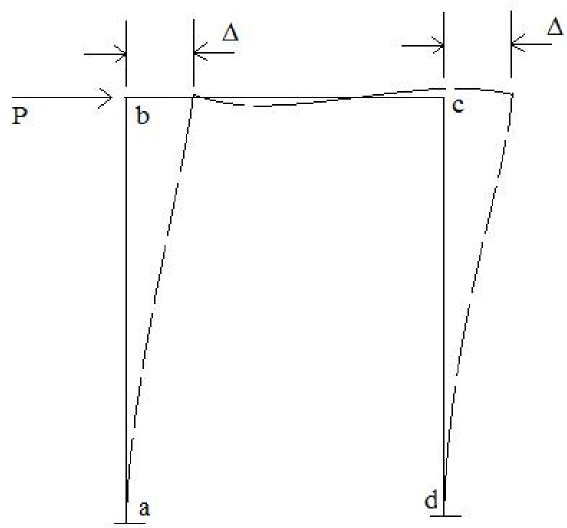
This is the same solution as the other methods, as expected.

Joint translations can often be related to each other through inspection. Sometimes joint translations relationships are *required* in order to solve for the joint moments.



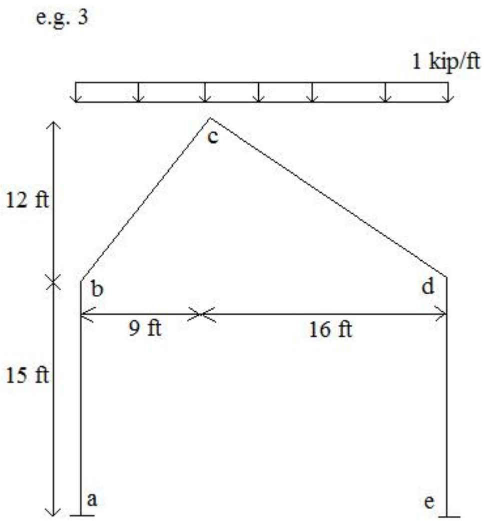
From the Law of Sines:

$$\frac{\Delta_1}{\sin\theta} = \frac{\Delta_2}{\sin(90-\theta)} = \frac{\Delta_3}{\sin 90}$$



From inspection, we can see that member bc will deform as shown, with $\theta_b = \theta_c$, because if an equal and opposite force is applied at c, then member bc (and the frame as a whole) must deform back to its original position (neglecting axial deformation in bc).

e.g. 3



$$(FEM)_{bc} = \frac{-(1)(9)^2}{12} = -6.750 kip*ft = -FEM_{cb}$$

$$(FEM)_{cd} = \frac{-(1)(16)^2}{12} = -21.333 kip*ft = -FEM_{dc}$$

(skipped work)

$$M_{ab} = 2E\frac{I}{15}(\theta_b - 3R_{ba}) \qquad (1)$$

$$M_{ba} = 2E\frac{I}{15}(2\theta_b - 3R_{ba}) \qquad (2)$$

$$M_{bc} = 2E\frac{I}{\sqrt{9^2+12^2}}(2\theta_b + \theta_c - 3R_{cb}) - 6.75 \qquad (3)$$

$$M_{cb} = 2E\frac{I}{\sqrt{9^2+12^2}}(2\theta_c + \theta_b - 3R_{cb}) + 6.75 \qquad (4)$$

$$M_{cd} = 2E\frac{I}{\sqrt{16^2+12^2}}(2\theta_c + \theta_d - 3R_{dc}) - 21.333 \qquad (5)$$

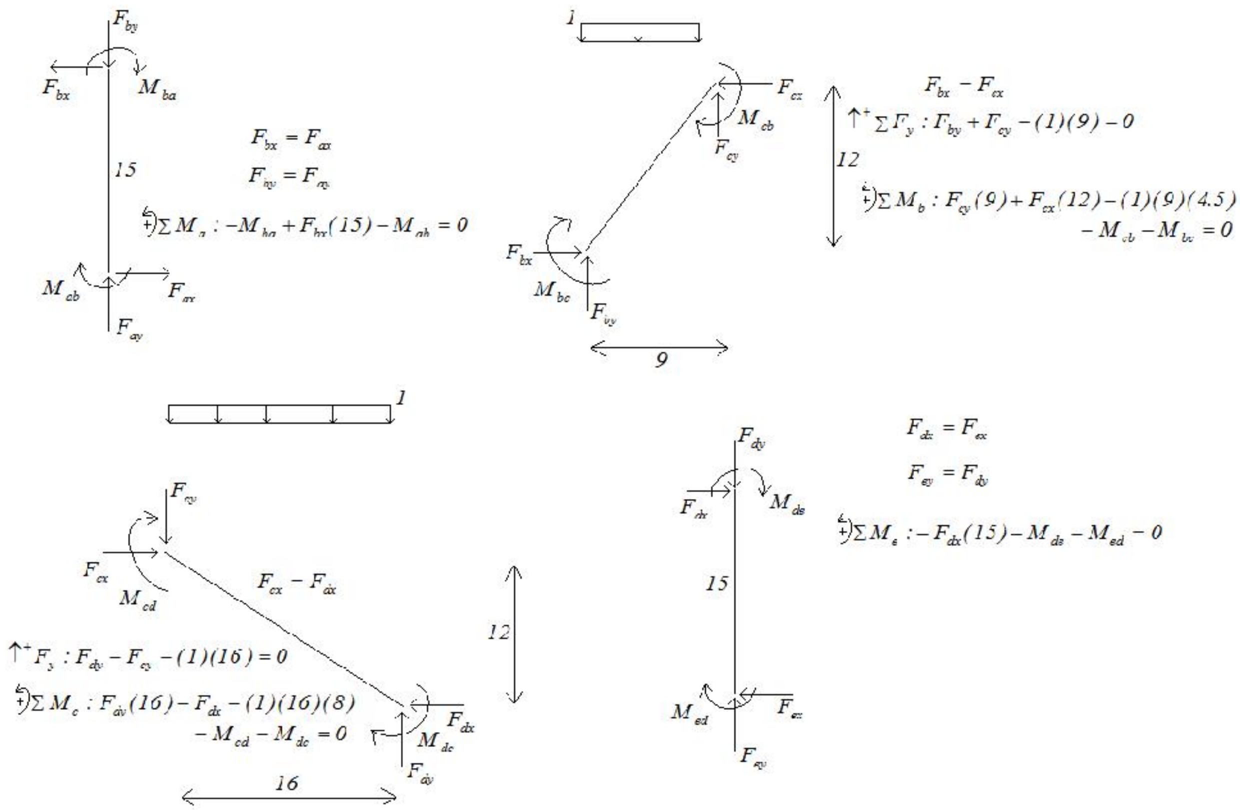
$$M_{dc} = 2E\frac{I}{\sqrt{16^2+12^2}}(2\theta_d + \theta_c - 3R_{dc}) + 21.333 \qquad (6)$$

$$M_{de} = 2E\frac{I}{15}(2\theta_d - 3R_{de}) \quad (7) \qquad M_{ed} = 2E\frac{I}{15}(\theta_d - 3R_{de}) \quad (8)$$

$$M_{ba} = -M_{bc} \quad (9) \qquad M_{cb} = -M_{cd} \quad (10) \qquad M_{dc} = -M_{de} \quad (11)$$

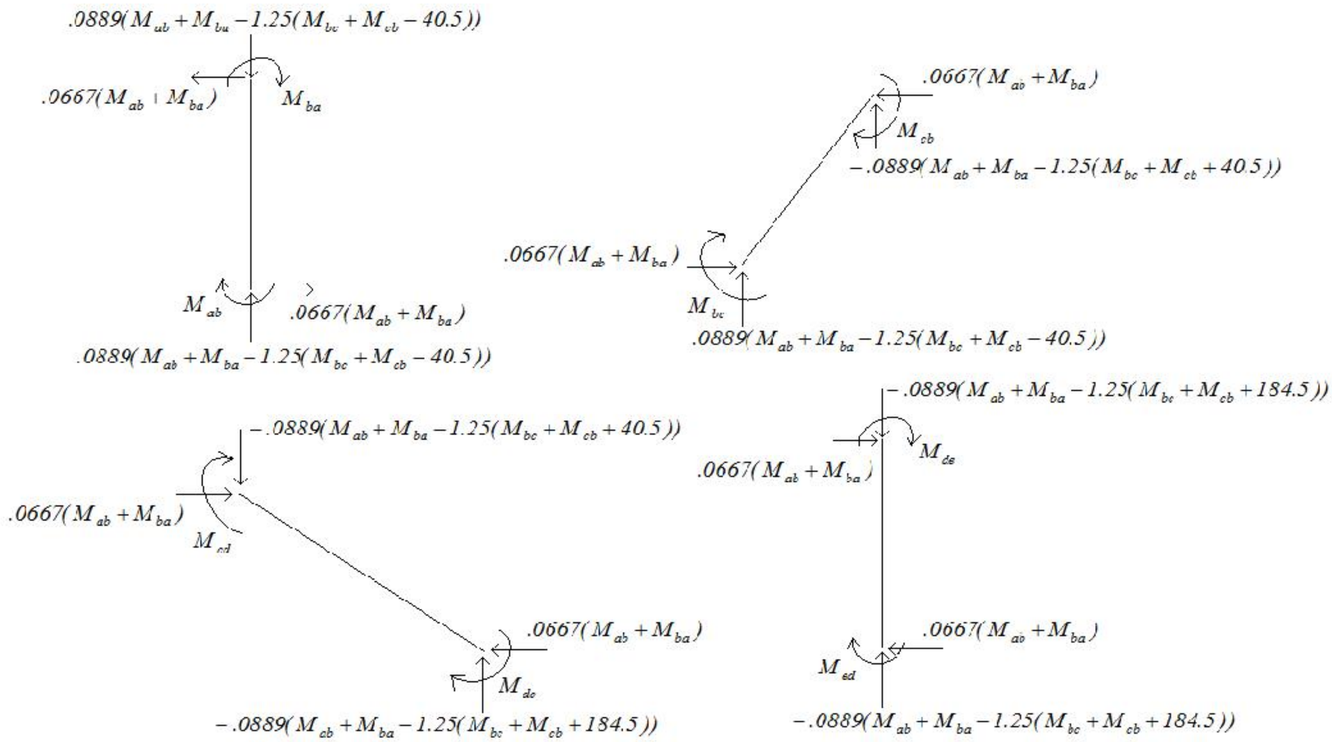
(15 unknowns; we need 4 more equations)

Additional equilibrium equations will introduce additional unknowns. In this case, as we will see below, introducing the additional equations of equilibrium (for each of the four segments of the structure) will only help to eliminate two unknowns.



10 unknowns here (moments treated as known)
12 equations.
Recall that we needed 14 equations, in this case, to solve for everything, since we needed to gain 4 additional from before.

While we're at it though, we can solve for joint forces in terms of moments (shown below) for later.



note: If we want to simply know the static indeterminacy of the frame, then we can concentrate solely on the picture above. We can consider the 18 unknowns (note that the forces are already assumed equal and opposite, as drawn, but the moments are not). The static indeterminacy will be the same, regardless of our method of approach.

12 equations + $(M_{ba} = -M_{bc}) + (M_{cb} = -M_{cd}) + (M_{dc} = -M_{de})$ *= 15 equations*

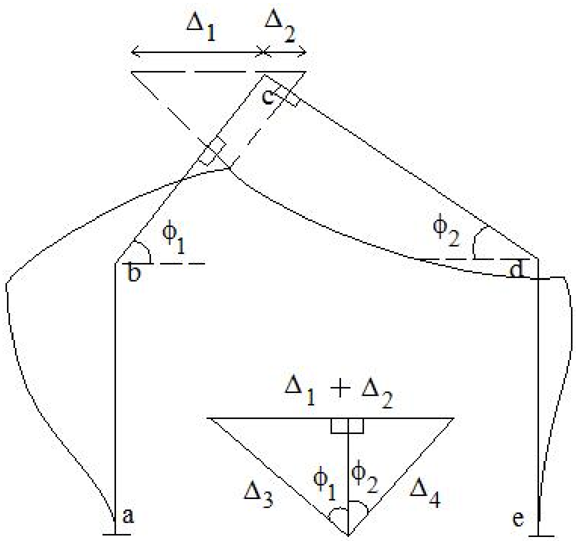
18 unknowns – 15 equations = 3 $\rightarrow$ *static indeterminacy of the frame*

(3b + r = 18, 3j = 15; 18 unknowns > 15 equations $\Rightarrow$ *indeterminate to 3*rd *degree)*

Returning to the slope-deflection equations, there are 15 unknowns and 13 equations (11 from before and the 2 additional from above). The additional two equations have not been explicitly written here, but they can come from $\sum M = 0\ (12), \sum F_y = 0\ (13)$ *for the frame as a whole, in terms of the joint moments shown above.*

We still need two more equations, but all equilibrium equations have now been used (if you try to use more equilibrium equations, solving will return a singular solution)

Our only option is to create two more equations by using deflection relationships.



Law of sines:

$$\frac{a_1 + a_2}{sin(¤_1 + ¤_2)} = \frac{a_3}{sin(90 - ¤_2)} = \frac{a_4}{sin(90 - ¤_1)}$$

$$¤_1 = tan^{-1}(\frac{12}{9}) = 53.13°$$

$$¤_2 = tan^{-1}(\frac{12}{16}) = 36.87°$$

$$\therefore$$

$$\frac{a_1 + a_2}{1} = \frac{a_3}{.8} = \frac{a_4}{.6}$$

$$a_3 = .8(a_1 + a_2) \qquad (14) \qquad\qquad a_4 = .6(a_1 + a_2) \qquad (15)$$

$$R_{ba} = \frac{-a_1}{15} \qquad (16) \qquad\qquad R_{cb} = \frac{a_3}{\sqrt{9^2 + 12^2}} \qquad (17)$$

$$R_{dc} = \frac{-a_4}{\sqrt{16^2 + 12^2}} \qquad (18) \qquad\qquad R_{de} = \frac{a_2}{15} \qquad (19)$$

Instead of 2 more equations, we obtained 6 more equations and 4 more unknowns (a_1, a_2, a_3, a_4). But this is fine – 19 equations, 19 unknowns

SOLVE

$$\Rightarrow -_b = \frac{.035(82.8)}{EI} \quad -_c = \frac{82.8}{EI} \quad -_d = \frac{-.755(82.8)}{EI} \text{ radians}$$

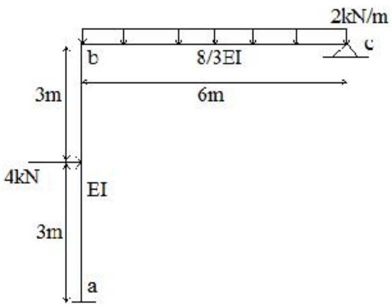
$$R_{ba} = \frac{-.77(82.8)}{EI} \quad R_{cb} = \frac{.95(82.8)}{EI} \quad R_{dc} = \frac{-.53(82.8)}{EI} \quad R_{de} = \frac{.414(82.8)}{EI} \text{ rad/ft}$$

$$a_1 = \frac{11.6(82.8)}{EI} \quad a_2 = \frac{6.2(82.8)}{EI} \quad a_3 = \frac{14.25(82.8)}{EI} \quad a_4 = \frac{10.7(82.8)}{EI} \text{ ft}$$

$$M_{ab} = 26.0 \quad M_{ba} = 26.4 \quad M_{bc} = -26.4 \quad M_{cb} = -2.2 \quad M_{cd} = 2.2 \quad M_{dc} = 30.4$$

$$M_{de} = -30.4 \quad M_{ed} = -22.0 \text{ kips * ft}$$

e.g. 4



(a)

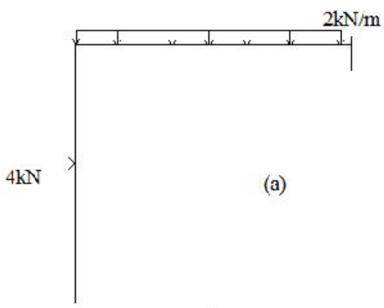
$$M_{ab} = \frac{2E(I)}{6}(0 + \bar{\ }_{b} - 0) - 3$$

$$M_{ba} = \frac{2E(I)}{6}(2\bar{\ }_{b} + 0 - 0) + 3$$

$$M_{bc} = \frac{2E(\frac{8}{3}I)}{6}(2\bar{\ }_{b} + 0 - 0) - 6$$

$$M_{cb} = \frac{2E(\frac{8}{3}I)}{6}(0 + \bar{\ }_{b} - 0) + 6$$

$$M_{ba} = -M_{bc}$$



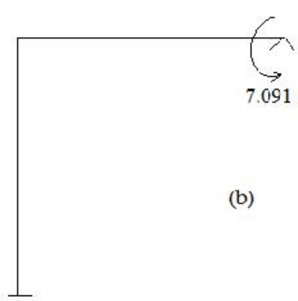
SOLVE $\Rightarrow$

$M_{ab} = -2.591 \quad M_{ba} = 3.818$

$M_{bc} = -3.818 \quad M_{cb} = 7.091$

+

(b)



$$M_{ab} = \frac{2E(I)}{6}(0 + \bar{\ }_{b})$$

$$M_{ba} = \frac{2E(I)}{6}(2\bar{\ }_{b})$$

$$M_{bc} = \frac{2E(\frac{8}{3}I)}{6}(2\bar{\ }_{b} + \bar{\ }_{c})$$

$$M_{cb} = \frac{2E(\frac{8}{3}I)}{6}(2\bar{\ }_{c} + \bar{\ }_{b})$$

$$M_{cb} = -7.091$$

$$M_{ba} = -M_{bc}$$

SOLVE $\Rightarrow M_{ab} = .591 \quad M_{ba} = 1.182 \quad M_{bc} = -1.182 \quad M_{cb} = -7.091$

(a) + (b) $\Rightarrow M_{ab} = -2 \quad M_{ba} = 5 \quad M_{bc} = -5 \quad M_{cb} = 0 \;\; kN * m$

OR

$$M_{ab} = \frac{2E(I)}{6}(0 + \bar{\ }_{b} - 0) - 3$$

$$M_{ba} = \frac{2E(I)}{6}(2\theta_b + 0 - 0) + 3$$
$$M_{bc} = \frac{2E(8/3\, I)}{6}(2\theta_b + \theta_c - 0) - 6$$
$$M_{cb} = \frac{2E(8/3\, I)}{6}(2\theta_c + \theta_b - 0) + 6$$
$$M_{ba} = -M_{bc}$$
$$M_{cb} = 0$$

$$SOLVE \Rightarrow M_{ab} = -2 \quad M_{ba} = 5 \quad M_{bc} = -5 \quad M_{cb} = 0 \;\; kN * m$$

$$(\theta_c = \frac{-4.875}{EI}, \theta_b = \frac{3}{EI}) \text{ radians}$$

The methods presented in this chapter on Classical Structural Analysis put into practice some of the ideas illustrated in statics and mechanics of materials, namely, that the distribution of forces in a frame is influenced by not only overall geometry but also by relative member stiffnesses. Courses on Matrix Structural Analysis (or FEA) present a more elegant method for solving for forces in statically indeterminate systems, utilizing linear and rotational deformation compatibility at every joint, for example, if beam and column "elements" are used. FEA is less intuitive, but is the method that is used by structural analysis computer software due to its versatility and computational efficiency when it comes to transforming analysis methods into computer algorithms. Just as courses on Numerical Methods should typically be taken after Differential Equations, Matrix Structural Analysis should be taken after Classical Structural Analysis. The methods for solving differential equations of motion for a simple system or for solving for the member forces in a simple frame, by hand, are entirely different, than the methods or algorithms that would be used by software to solve more complex dynamic or static problems. Courses on Matrix Structural Analysis and the related field of Finite Element Analysis are typically taken at the graduate level.

Works Cited

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Trifunac, Mihailo. Lecturer. University of Southern California. CE358. Fall 2005.

Appendix

Appendix B: FEMs