Castigliano's second theorem

"the first partial derivative of the total strain energy of the structure with respect to one of the applied actions gives the displacement along that action"

$\Delta_P =$ corresponding displacement (deflection or rotation) along P,
where P = particular force or couple

$$\Delta_P = \frac{dW}{dP}$$

For a loaded beam, total strain energy $W = \int_0^L \frac{M^2 dx}{2EI}$ (derived in the section titled "External work and internal work")

For a loaded truss, total strain energy $W = \sum \frac{S^2 L}{2AE}$

Note: $M = M_1 + M_2 = m_1 P_1 + m_2 P_2$, where M = bending moment at any section
M_1 = moment at any section due to load P_1
m_1 = bending moment at any section due to a unit load in place of P_1

The fact that M_1 can be represented by the product of m_1 and P_1 (and the same for M_2, m_2, and P_2) and the fact that M can be represented by the sum of M_1 and M_2, are both important principles made possible by the principle of superposition.

Using the Chain Rule for derivatives (skipped work) ;

Castigliano's Theorem for beams :

$$\boldsymbol{\Delta_1 = \int_0^L \frac{M(\frac{dM}{dP_1})}{EI} dx}$$

Castigliano's Theorem for trusses :

$$\boldsymbol{\Delta_1 = \sum \frac{S(\frac{dS}{dP_1})L}{AE}}$$

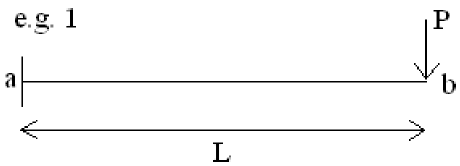
We can easily show that Castigliano's Theorem and the unit load method are really one in the same:

$$\Delta_1 = \frac{dW}{dP_1} = \frac{d}{dP_1}\int_0^L \frac{(m_1 P_1 + m_2 P_2)^2}{2EI} dx = \int_0^L \frac{2(m_1 P_1 + m_2 P_2)(m_1 + 0)}{2EI} dx = \int_0^L \frac{M m_1}{EI} dx$$

Similarly, $\Delta_2 = \int_0^L \frac{Mm_2}{EI} dx$, as expected.

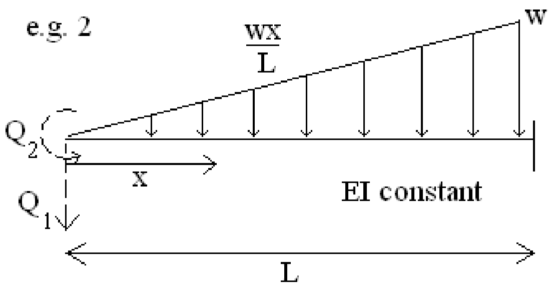
note: Theorem applies for θ calculations as well.

e.g. 1



$$\Delta_P = \Delta_b = \int_0^L \frac{(-Px)(-x)}{EI} dx = \boldsymbol{\frac{PL^3}{3EI}}$$

e.g. 2
Castigliano's Theorem clearly works with multiple loads and can also work at points where a load is not present, by placing an imaginary load P_1 at the point of interest and setting up the equation for Δ_1 in terms of P_1. Then, set $P_1 = 0$ either before or after integrating/summing.



$$w = 1.5\, {}^{kN}\!/\!_{m}$$

$$L = 5m$$

$$E = 20000 \frac{kN}{cm^2}$$

$$I = 12000\, cm^4$$

$$M = -Q_1 x - Q_2 - \frac{wx^3}{6L}$$

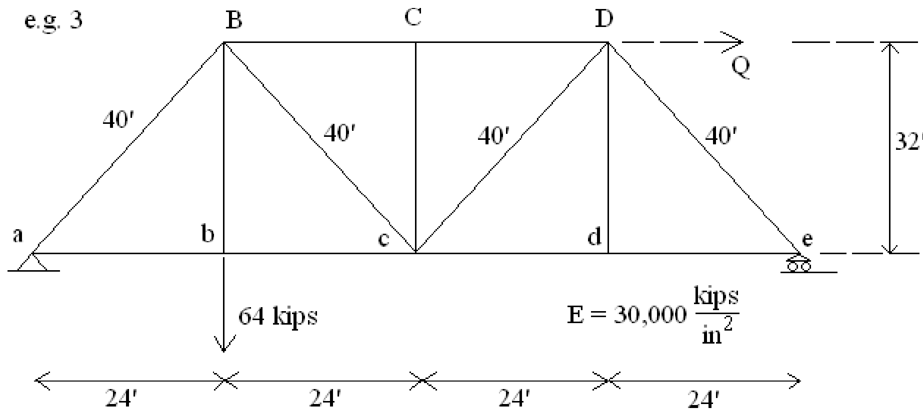
$$\frac{dM}{dQ_1} = -x \text{ and } \frac{dM}{dQ_2} = -1$$

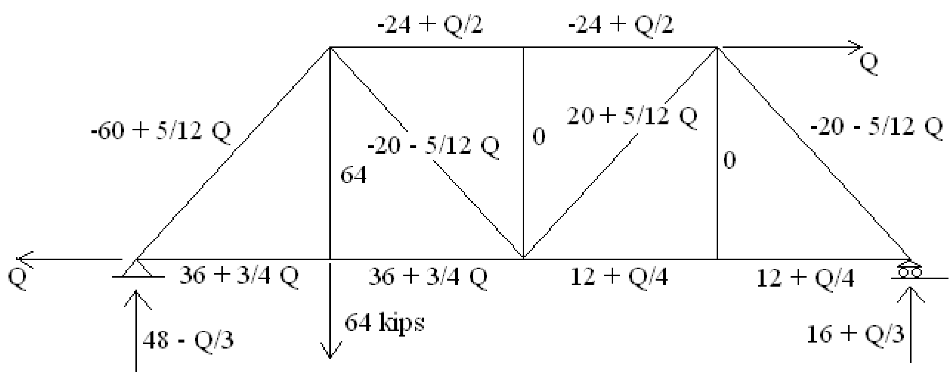
$$\Delta_a = \int_0^L \frac{M(\frac{dM}{dQ_1})}{EI} dx = \int_0^L \frac{(-\frac{wx^3}{6L})(-x)}{EI} dx = \frac{wL^4}{30EI} = \mathbf{.13\ cm\ (down)}$$

we're using $M = -\frac{wx^3}{6L}$ *since* $Q_1 = Q_2 = 0$

$$\theta_a = \int_0^L \frac{(-\frac{wx^3}{6L})(-1)}{EI} dx = \frac{wL^3}{24EI} = \mathbf{.0003\ rad\ (ccw)}$$

e.g. 3   (same truss as the truss example in previous section)





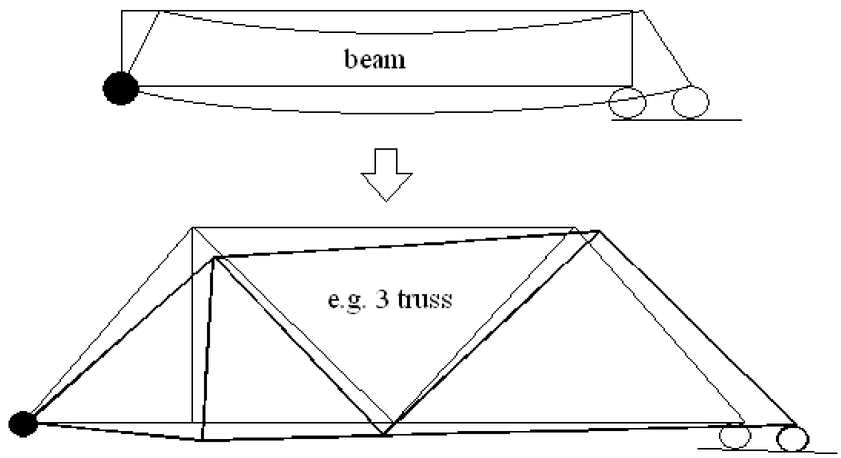
member	S	dS/dQ	$\frac{S(dS/dQ)L}{A}$	
ab	36 + 3/4 Q	3/4	27	(setting Q = 0)
bc	36 + 3/4 Q	3/4	27	
cd	12 + 1/4 Q	1/4	3	
de	12 + 1/4 Q	1/4	3	
BC	-24 + 1/2 Q	1/2	-12	
CD	-24 + 1/2 Q	1/2	-12	
aB	-60 + 5/12 Q	5/12	-25	
Bb	64	0	0	
Bc	-20 - 5/12 Q	-5/12	8.333	
Cc	0	0	0	
cD	20 + 5/12 Q	5/12	8.333	
Dd	0	0	0	
De	-20 - 5/12 Q	-5/12	8.333	
		$\sum$	36	

$$\Delta = \sum \frac{S(\,{}^{dS}\!/\!_{dQ})L}{AE} = \frac{36}{30000}$$

= ***.0012 ft (right)***

note: This result may seem odd. Since the top chord is in compression, one might expect to see movement at joint D to the left.

The fact that joint D moves to the right is entirely due to the fact that our left pin is immovable and our right roller slides to the right. This is, in fact, exactly how bridges are constructed. Bridge structures are exposed to the elements, so the bridge sits on a roller to allow for expanding/contracting due to temperature fluctuations. (See the diagrams below) So, joint D <u>would</u> move to the right, as we've found.



For beams in buildings, we should remember that the typical left pin/right roller is <u>not</u> the reality. Typical beams in buildings do not actually sit on rollers, but are pin/pin. The left pin/right roller idealization for typical beams yields exactly the same results as a pin/pin, however, because of all of our beam assumptions. These assumptions include assuming the supports are at the neutral axis (a perfectly valid assumption for typical beams) which PREVENTS any horizontal reaction from creating a moment due to eccentricity, the idea of conservative force which PREVENTS any horizontal reaction from creating a moment due to the beam's curvature, and the neglecting of axial deformation. All of these assumptions result in pin/pin and pin/roller yielding identical results. I.e. we can use the methods in the following sections for finding redundant forces and we would see that the horizontal force is zero for the case of a pin/pin beam with vertical loads, as would obviously be the case for a pin/roller. We can then use formulas we already know from this and previous sections for finding deflections and we would see that the vertical deflections are also identical for pin/pin versus pin/roller. Does this seem reasonable?

A beam that is pin-pin is restrained against horizontal motion, whereas a beam that is idealized as pin/roller or roller/roller is not restrained. Intuitively, this makes a difference. Intuitively, if subjected only to vertical force, the beam that is pin/pin will still have horizontal reactions whereas the beam that is roller/roller or pin/roller will not have horizontal reactions, since it is free to move. However, these horizontal reactions are small and, intuitively, are small enough to be neglected. In practice, the typical beams analyzed in this manner (simple supports) are checked for vertical force and vertical deflections, which intuitively would NOT be SIGNIFICANTLY affected by the pin/pin restraints. In practice, the typical beams are completely ignored when lateral drifts are checked. So, luckily, we don't need to conclude that all beam-related formulas up to this point are false and develop new formulas. The fact that pin/pin or pin/roller makes no difference for vertical forces, horizontal forces, and vertical deflections of our typical beams seems reasonable, because in reality the differences would in fact be very small.

Unlike typical beams, the supports of moment frames and braced frames and trusses should always be modeled exactly how they are actually built. As we will see in the following sections, when we consider a <u>system</u> of beams or bars, such as a moment frame, horizontal reactions <u>will</u>

be developed from vertical forces. We will also consider a pin/pin truss. Unlike a beam, a truss that is pin/pin is different than a truss that is pin/roller. A truss will have different forces and deflections, depending on the support configuration.

Works Cited

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