Appendix A

Math derivations

A.1 Transpose of tensor product

We'd like to prove that $\mathbf{A}^T = \mathbf{C}^T \cdot \mathbf{B}^T$, if $\mathbf{A}$ is defined as $\mathbf{A} = \mathbf{B} \cdot \mathbf{C}$.

In other words, we will show that $(\mathbf{B} \cdot \mathbf{C})^T = \mathbf{C}^T \cdot \mathbf{B}^T$

We can start with a particular definition of transpose, namely,

$$\mathbf{b} \cdot \mathbf{A}^T \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{b} \tag{A.1}$$

In eq. (A.1), vectors $\mathbf{a}$ and $\mathbf{b}$ are arbitrary.

Using this definition of transpose (eq. (A.1)), and then the associative law for vector and tensor dot products,

$\mathbf{a} \cdot (\mathbf{B} \cdot \mathbf{C})^T \cdot \mathbf{b} = \mathbf{b} \cdot (\mathbf{B} \cdot \mathbf{C}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{B} \cdot (\mathbf{C} \cdot \mathbf{a})$

By invoking the definition of transpose, and then the commutative law of addition for vector dot products, and then the associative law for products,

$\mathbf{a} \cdot (\mathbf{B} \cdot \mathbf{C})^T \cdot \mathbf{b} = (\mathbf{C} \cdot \mathbf{a}) \cdot \mathbf{B}^T \cdot \mathbf{b} = \mathbf{B}^T \cdot \mathbf{b} \cdot (\mathbf{C} \cdot \mathbf{a}) = (\mathbf{B}^T \cdot \mathbf{b}) \cdot \mathbf{C} \cdot \mathbf{a}$

Now, we can once again invoke the definition of transpose, and then the associative law for vector and tensor dot products:

$\mathbf{a} \cdot (\mathbf{B} \cdot \mathbf{C})^T \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{C}^T \cdot (\mathbf{B}^T \cdot \mathbf{b}) = \mathbf{a} \cdot (\mathbf{C}^T \cdot \mathbf{B}^T) \cdot \mathbf{b}$

Therefore, $(\mathbf{B} \cdot \mathbf{C})^T = \mathbf{C}^T \cdot \mathbf{B}^T$

A.2 Skew tensor

In order to show that $\mathbf{W}$ is "skew," we need to show that $\mathbf{W} = -\mathbf{W}^T$. It is sufficient to show that $\mathbf{L} - \mathbf{L}^T = -(\mathbf{L} - \mathbf{L}^T)^T$.

Similar to the proof shown in Appendix A.1, we will start with arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$.

Consider that the definition of transpose (eq. (A.1)) states that:

$$\mathbf{a} \cdot (\mathbf{L} - \mathbf{L}^T)^T \cdot \mathbf{b} = \mathbf{b} \cdot (\mathbf{L} - \mathbf{L}^T) \cdot \mathbf{a}$$

We can expand the last term and again invoke the definition of transpose:

$$\mathbf{a} \cdot (\mathbf{L} - \mathbf{L}^T)^T \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{L} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{L}^T \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{L}^T \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{L} \cdot \mathbf{b}$$

Finally, we see that the last term can be reduced:

$$\mathbf{a} \cdot (\mathbf{L} - \mathbf{L}^T)^T \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{L}^T - \mathbf{L}) \cdot \mathbf{b} = \mathbf{a} \cdot -(\mathbf{L} - \mathbf{L}^T) \cdot \mathbf{b}$$

This proves that $\mathbf{W}$ is indeed anti-symmetric or "skew."

A.3 Orthogonal tensor

Orthogonality of $\mathbf{R}$ will be proven if we can show that the product of $\mathbf{R}$ with its transpose yields the identity tensor, $\mathbf{I}$.

Since $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$, we know that $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$

So, $\mathbf{R}^T \cdot \mathbf{R} = (\mathbf{F} \cdot \mathbf{U}^{-1})^T \cdot (\mathbf{R} \cdot \mathbf{U}^{-1})$

From Appendix A.1, we know that $(\mathbf{F} \cdot \mathbf{U}^{-1})^T = (\mathbf{U}^{-1})^T \cdot \mathbf{F}^T$

Thus,

$$\mathbf{R}^T \cdot \mathbf{R} = (\mathbf{U}^{-1})^T \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{U}^{-1} \cdot \underbrace{\mathbf{F}^T \cdot \mathbf{F}}_{\mathbf{C}} \cdot \mathbf{U}^{-1} = \mathbf{U}^{-1} \cdot \mathbf{U}^2 \cdot \mathbf{U}^{-1}$$

Since $\mathbf{U}^{-1} \cdot \mathbf{U}^2 \cdot \mathbf{U}^{-1}$ reduces to $\mathbf{I}$, we have the desired result.

Appendix B

Stress derivations

B.1 Physical interpretation of $\boldsymbol{\sigma}$

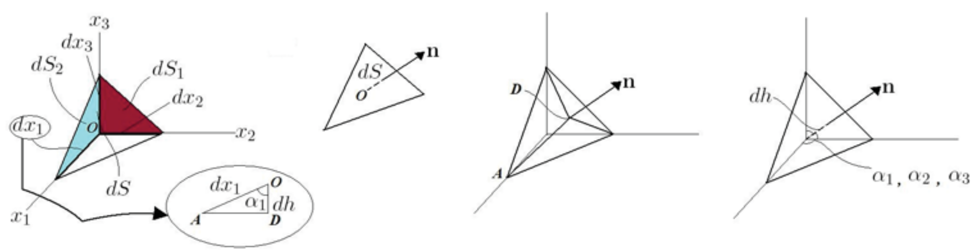

Figure B.1: Stress wedge

Consider the stress "wedge," or "Cauchy tetrahedron," depicted in Fig. B.1, where we would like to know the stresses acting on the plane defined by a unit vector $\mathbf{n}$:

$\mathbf{n} = n_i \mathbf{e_i}$

Noting the triangle AOD in Fig. B.1, with angle α_1:

$\underbrace{cos\alpha_1}_{n_1} = \frac{dh}{dx_1}$

We note here that $cos\alpha_1$ is the component of $\mathbf{n}$ in the "1" direction.

So, $dx_i = \frac{dh}{n_i}$

Now, $dV = \cancel{\tfrac{1}{3}} dS \cdot dh = \cancel{\tfrac{1}{3}} dx_1 dS_1 = \cancel{\tfrac{1}{3}} dx_2 dS_2 = \cancel{\tfrac{1}{3}} dx_3 dS_3$

The above expression for dV may be hard to visualize, but we're essentially multiplying a plane by a distance, analogous to $V = Ah$ for a cylinder.

Solving for, say, $dS_1 \rightarrow dS_1 = \frac{dSdh}{dx_1}$... *etc.*.
In general, $dS_i = dS\frac{dh}{dx_i}$

We note also that this above expression can be re-written, since $n_i = \frac{dh}{dx_i}$:

$$dS_i = dSn_i \tag{B.1}$$

i.e. If n_1 is small, then the surface dS_1 defined by $\mathbf{n}$ and shown in Fig. B.1, is small (dx_1 is large).

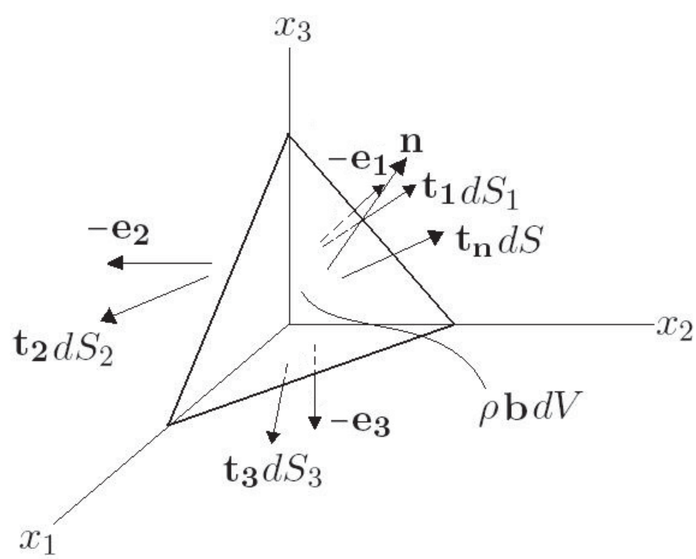

Figure B.2: Stresses

If $\rho\mathbf{b}dV$ is the "body force" (force due to gravity, for example), and $\mathbf{t}$ is stress (Fig. B.2), then:

$\sum F : \mathbf{t_n}dS - \sum \mathbf{t_i}dS_i + \rho\mathbf{b}dV = \rho dV\frac{d\mathbf{v}}{dt}$,
where $\rho dV\frac{d\mathbf{v}}{dt}$ is essentially $mass * acceleration$

Take $dh \longrightarrow 0$ since we want the stresses at a point:

Divide through by dS and note that $\frac{dV}{dS} \longrightarrow 0$ as $dh \longrightarrow 0 \longrightarrow \mathbf{t_n} = \sum \mathbf{t_i} \frac{dS_i}{dS}$

But, it was previously shown (eq. (B.1)) that $\frac{dS_i}{dS} = n_i$.

So, $\mathbf{t_n} = \sum \mathbf{t_i} n_i$, where $\mathbf{t_i} = \mathbf{n_i} t_i$, as depicted in Fig. B.2.

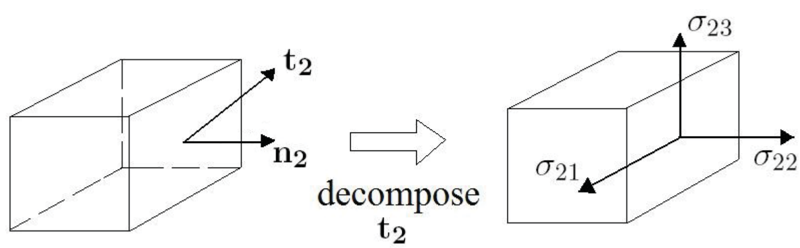

Figure B.3: Stress vector components

Consider, for example, $\mathbf{t_2}$, shown in Fig. B.3:

note: Normal and shear stress magnitudes are commonly denoted by σ.

So, $\mathbf{t_2} = [\sigma_{21}, \sigma_{22}, \sigma_{23}]$
Here, the first subscript may be thought of as face "2" (Fig. B.2) and the second subscript can be thought of as the direction of stress on that particular face.

$\mathbf{t_2} = \sigma_{2i} \mathbf{e_i}$

In general, $\mathbf{t_i} = \sigma_{ij} \mathbf{e_j}$

We know that $\mathbf{t_n} = \sum \mathbf{t_i} n_i$

Substituting, we get $\mathbf{t_n} = \sum \sigma_{ij} \mathbf{e_j} n_i = \sigma_{ij} \mathbf{e_j} n_i$ (index notation)

If $\mathbf{n} = n_i \mathbf{e_i}$ and $\overbrace{\sigma_{ij} \mathbf{e_i} \otimes \mathbf{e_j}}^{\text{stress tensor}}$, then $\mathbf{t_n} = \sigma_{ij} \mathbf{e_j} n_i$ is matrix-vector multiplication so long as $\boldsymbol{\sigma}$ is a symmetric tensor (recall that $\mathbf{A} \cdot \mathbf{b} = A_{ij} \mathbf{e_i} b_j$ is the expression that defines matrix-vector multiplication, regardless of symmetry).

$$\mathbf{t_n} = \boldsymbol{\sigma} \cdot \mathbf{n} \tag{B.2}$$

Since we are really concerned with points in a body rather than volumes (recall that we took $dh \longrightarrow 0$ earlier), the physical meaning of eq. (B.2) is essentially as follows: if we know the normal and shear stresses at a particular point in a body (with respect to a particular bases), then we can find the stresses in any direction (or at any angle).

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{B.3}$$

The most common proof of eq. (B.3) involves summing moments (*i.e.* "conservation of angular momentum") and since we skipped how to do cross products, we'll skip this proof (the complete proof can be found in [2]).

B.2 Equation of Motion

Let's start with:

$$\int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS + \int_V \rho \mathbf{b} dV = \frac{d}{dt} \int_V \rho \mathbf{v} dV \tag{B.4}$$

Our goal is to arrive at the following result:

$$\int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV + \int_V \rho b_i dV = \int_V \rho \frac{dv_i}{dt} dV \tag{B.5}$$

In order to accomplish this, we must make several observations. First, we need the divergence theorem for second-order tensors.

For vectors, the divergence theorem can be written:

$$\int_S \mathbf{u} \cdot \mathbf{n} dS = \int_V div(\mathbf{u}) dV \tag{B.6}$$

In eq. (B.6), $div(\mathbf{u}) = \frac{\partial u_i}{\partial x_i}$

For tensors, we have:

$$\int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_V div(\boldsymbol{\sigma}) dV \tag{B.7}$$

In eq. (B.7), $div(\boldsymbol{\sigma}) = \frac{\partial \sigma_{ij}}{\partial x_j}$

We can derive eq. (B.7) as follows:
Let $\mathbf{b}$ be an arbitrary vector:
$\mathbf{b} \cdot \int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_S \mathbf{b} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS$
Using the definition of transpose (Appendix A.1) and the associative law for vectors, we get:
$\int_S (\boldsymbol{\sigma}^T \cdot \mathbf{b}) \cdot \mathbf{n} dS$
From the divergence theorem for vectors (eq. (B.6)), we get:
$\int_S (\boldsymbol{\sigma}^T \cdot \mathbf{b}) \cdot \mathbf{n} dS = \int_V div(\boldsymbol{\sigma}^T \cdot \mathbf{b}) dV$
Note that: $div(\boldsymbol{\sigma}^T \cdot \mathbf{b}) = \frac{\partial}{\partial x_i}(\boldsymbol{\sigma}^T \cdot \mathbf{b}) = \frac{\partial \sigma_{ki}}{\partial x_i} b_k + \sigma_{ki} \cancel{\frac{\partial b_k}{\partial x_i}} = div(\boldsymbol{\sigma}^T) \cdot \mathbf{b}$ where the slashed term is zero.
To complete the derivation, note that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ from Appendix B.1. Additionally, noting the associative law of vectors, we get the desired result, namely:
$\cancel{\mathbf{b}} \cdot \int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_V \cancel{\mathbf{b}} \cdot div(\boldsymbol{\sigma}) dV = \cancel{\mathbf{b}} \cdot \int_V div(\boldsymbol{\sigma}) dV$

The first term in eq. (B.7) is now understood, but to derive the last term in eq. (B.7), we need to make two additional observations.

First, we know from a previous derivation that:

$$dV = det(\mathbf{F}) dV_0 \tag{B.8}$$

We can similarly express ρ in terms of ρ_0 by noting from the conservation of mass, that $\int_V \rho dV = \int_{V_0} \rho_0 dV_0$

Substituting eq. (B.8), we get $\int_V \rho dV = \int_{V_0} \rho det(\mathbf{F}) dV_0 = \int_{V_0} \rho_0 dV_0$

Thus, we have $\int_{V_0} (\rho det\mathbf{F} - \rho_0) dV_0 = 0$, which is true for any arbitrary V_0. So,

$$\rho = \frac{\rho_0}{det\mathbf{F}} \tag{B.9}$$

Finally, substituting eq. (B.8) and eq. (B.9) into the last term in eq. (B.4), we get:

$\frac{d}{dt} \int_V \rho \mathbf{v} dV = \int_{V_0} \frac{d}{dt} \frac{\rho_0}{\cancel{det(\mathbf{F})}} \mathbf{v} \cancel{det(\mathbf{F})} dV_0$

Since ρ_0 and V_0 are constant with time, we have:

$$\frac{d}{dt}\int_V \rho \mathbf{v} dV = \int_{V_0} \rho_0 \frac{d\mathbf{v}}{dt} dV_0 \tag{B.10}$$

The final step is to substitute $\rho_0 = \rho det\mathbf{F}$ and $dV_0 = \frac{dV}{det(\mathbf{F})}$ into eq. (B.10). We then arrive at the expected result:

$$\frac{d}{dt}\int_V \rho \mathbf{v} dV = \int_V \rho \frac{d\mathbf{v}}{dt} dV \tag{B.11}$$

Substituting eq. (B.7) and eq. (B.11) into eq. (B.4), we get the desired result (eq. (B.5)).

Appendix C

Hyperelastic derivations

C.1 Proof of $\boldsymbol{\sigma} = f(\mathbf{B})$

Assume $\boldsymbol{\sigma}$ is a function of $\mathbf{F}$

We know from the Chapter 5, that for a superimposed strain:
$\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}$
$\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T$

So, $\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T = f(\mathbf{Q} \cdot \mathbf{F})$

From polar decomposition, we know $\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U}$

We will first consider the latter form of $\mathbf{F}$. Let's take $\mathbf{Q} = \mathbf{R}^T$, where $\mathbf{Q}$ can be any orthogonal tensor. To see why we pick $\mathbf{Q} = \mathbf{R}^T$, recall that $\mathbf{U}$ is defined in "material" coordinates, and so $\mathbf{U}$ is, accordingly, invariant to any rigid body rotation. $\boldsymbol{\sigma}$ is defined in spatial coordinates and is <u>not</u> invariant to rotation. So, naturally, $\boldsymbol{\sigma} = f(\mathbf{F}) = f(\mathbf{R} \cdot \mathbf{U})$ is a function of both $\mathbf{U}$ and $\mathbf{R}$, rather than just $\mathbf{U}$. We can take $\boldsymbol{\sigma}^{*} = f(\mathbf{F}^*)$ and set $\mathbf{Q}$ equal to $\mathbf{R}^T$, for convenience.

$\mathbf{R}^T \cdot \boldsymbol{\sigma} \cdot \mathbf{R} = f(\mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U}) \longrightarrow \boldsymbol{\sigma} = \mathbf{R} \cdot f(\underbrace{\mathbf{R}^T \cdot \mathbf{R}}_{\mathbf{I}} \cdot \mathbf{U}) \cdot \mathbf{R}^T$

$\boldsymbol{\sigma} = \mathbf{R} \cdot f(\mathbf{U}) \cdot \mathbf{R}^T$

Recall that $\mathbf{C} = \mathbf{U}^2$

$$\boldsymbol{\sigma} = \mathbf{R} \cdot g(\mathbf{C}) \cdot \mathbf{R}^T \qquad \text{(C.1)}$$

$\boxed{\text{note: } \hat{\boldsymbol{\sigma}} = g(\mathbf{C})}$

Perhaps eq. (C.1) seems obvious, since $\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T$ and $\mathbf{C}^* = \mathbf{C}$, but we started with $\boldsymbol{\sigma} = f(\mathbf{F})$ for completeness.

Now, let's apply $\mathbf{Q}$ before deformation:

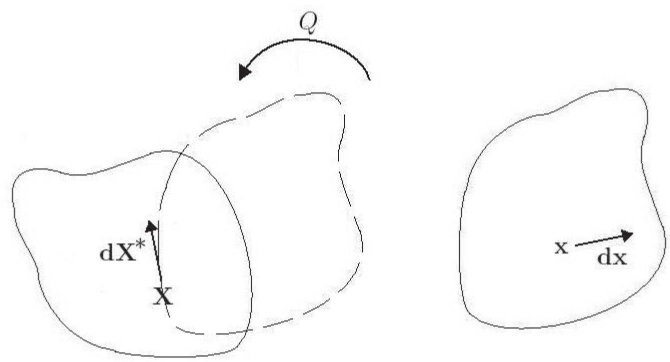

Figure C.1: Rigid body rotation applied prior to deformation

We can define $\mathbf{dX}^*$ as follows:

$\mathbf{dX}^* = \mathbf{Q} \cdot \mathbf{dX}$

Recall that last time we superimposed a rigid body rotation on $\mathbf{dx}$, which resulted in $\mathbf{dx}^* = \mathbf{Q} \cdot \mathbf{dx}$.

This time, we want $\mathbf{dx}^* = \mathbf{dx}$

We can see from Fig. C.2, that we need $\mathbf{F}^*$ to be a function of $\mathbf{Q}^T$

We can show that this is indeed the case as follows:

$\mathbf{dx}^* = \mathbf{dx} = \mathbf{F}^* \cdot \underbrace{\mathbf{Q} \cdot \mathbf{dX}}_{\mathbf{dX}^*}$

Since $\mathbf{dx} = \mathbf{F} \cdot \mathbf{dX}$, we find that $\mathbf{F} = \mathbf{F}^* \cdot \mathbf{Q}$, which yields:

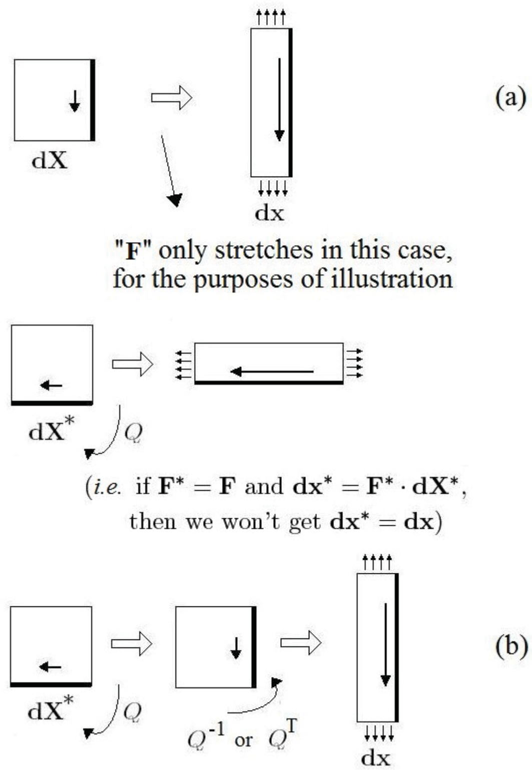

Figure C.2: Rigid body rotation applied prior to deformation

$$\mathbf{F}^* = \mathbf{F} \cdot \mathbf{Q}^T \tag{C.2}$$

Eq. (C.2) is what we wanted to find, and it should be expected. Recall that $\mathbf{V} \cdot \mathbf{R}$ is physically understood to be a rotation, $\mathbf{R}$, followed by a deformation (axial strains and shear), $\mathbf{V}$. Thus, $\mathbf{F}^* = \mathbf{F} \cdot \mathbf{Q}^T$ is an explicit rigid body rotation, $\mathbf{Q}^T$, followed by the total deformation + rotation, $\mathbf{F}$. In other words, the rotation, $\mathbf{Q}^T$, is applied to the initial configuration $\mathbf{dX}$.

Our new definition of $\mathbf{F}^*$ is difficult to physically interpret from a Lagrangian point-of-view, but we will use it in order to show that $\boldsymbol{\sigma} = f(\mathbf{V})$, as follows.

We know that $\boldsymbol{\sigma}^{\boldsymbol{*}} = f(\mathbf{F}^*)$, where our new definition of "$*$" requires that $\boldsymbol{\sigma}^{\boldsymbol{*}} = \boldsymbol{\sigma}$ and $\mathbf{F}^* = \mathbf{F} \cdot \mathbf{Q}^T$

So, $\boldsymbol{\sigma} = f(\mathbf{F} \cdot \mathbf{Q}^T)$

We know that $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$

Substituting $\longrightarrow \boldsymbol{\sigma} = f(\mathbf{V} \cdot \mathbf{R} \cdot \mathbf{Q}^T)$

Again, since we are simply applying our "$*$" operator to both $\boldsymbol{\sigma}$ and $f(\mathbf{F})$, we can take $\mathbf{Q}$ to be anything we want, as doing so is analogous to operating on both sides of any ordinary equation. Here, we can take $\mathbf{Q}$ to be $\mathbf{R}$.

This allows us to directly arrive at the desired result:

$$\boldsymbol{\sigma} = f(\mathbf{V}) \tag{C.3}$$

The result (eq. (C.3)) is expected since it was shown in Chapter 5 that $\boldsymbol{\sigma}$ and $\mathbf{V}$ are work-conjugate. Recall also that $\mathbf{V}^2 = \mathbf{B}$.

C.2 Derivation: $\frac{dI_B}{d\mathbf{B}}$, $\frac{dII_B}{d\mathbf{B}}$, $\frac{dIII_B}{d\mathbf{B}}$

$$\text{Since } I_B = tr\mathbf{B} = B_{kk}, \quad \frac{dI_B}{d\mathbf{B}} = \frac{\partial B_{nn}}{\partial B_{kl}}\mathbf{e_k}\mathbf{e_l} = \delta_{nk}\delta_{nl}\mathbf{e_k}\mathbf{e_l} \tag{C.4}$$

$$\frac{dII_B}{d\mathbf{B}} = \frac{d\left[\frac{1}{2}[(tr\mathbf{B})^2 - tr(\mathbf{B^2})]\right]}{d\mathbf{B}} = \frac{1}{2}[\underbrace{2tr\mathbf{B}\frac{d(tr\mathbf{B})}{d\mathbf{B}}}_{\text{chain rule}} - \frac{d(tr(\mathbf{B^2}))}{d\mathbf{B}}]$$

$$= \frac{1}{2}[2tr(\mathbf{B})\mathbf{I} - \underbrace{\frac{d(tr(\mathbf{B^2}))}{d\mathbf{B}}}_{\text{see below}}]$$

$$\frac{d(tr\mathbf{B}^2)}{d\mathbf{B}} = \frac{d(B_{ij}B_{ji})}{d\mathbf{B}} = \frac{\partial(B_{ij}B_{ji})}{\partial B_{mn}}\mathbf{e_n}\mathbf{e_m} = \underbrace{\frac{\partial B_{ij}}{\partial B_{mn}}B_{ji}\mathbf{e_n}\mathbf{e_m} + \frac{\partial B_{ji}}{\partial B_{mn}}}_{\text{product rule}}B_{ij}\mathbf{e_n}\mathbf{e_m}$$

$$= \delta_{im}\delta_{jn}B_{ji}\mathbf{e_n}\mathbf{e_m} + \delta_{jm}\delta_{in}B_{ij}\mathbf{e_n}\mathbf{e_m} = B_{ji}\mathbf{e_j}\mathbf{e_i} + B_{ij}\mathbf{e_i}\mathbf{e_j} = 2B_{ij}\mathbf{e_i}\mathbf{e_j}$$

$$\text{So, } \frac{dII_B}{d\mathbf{B}} = \frac{1}{2}[2tr(\mathbf{B})\mathbf{I} - 2B_{ji}\mathbf{e_i}\mathbf{e_j}] = I_B\mathbf{I} - \mathbf{B} \tag{C.5}$$

We know $III_B = det\mathbf{B}$; but we need a better expression for III_B before we derive $\frac{dIII_b}{d\mathbf{B}}$.
We know from the Cayley-Hamilton Theorem: $\mathbf{B}^3 - I_B\mathbf{B}^2 + II_B\mathbf{B} - III_B\mathbf{I} = 0$;

$tr(\mathbf{B}^3 - I_B\mathbf{B}^2 + II_B\mathbf{B} - III_B\mathbf{I}) = tr(0)$

$tr(\mathbf{B}^3) - \underbrace{tr(I_B\mathbf{B}^2)}_{I_B tr(\mathbf{B}^2)} + \underbrace{tr(II_B\mathbf{B})}_{II_B tr\mathbf{B}} - \underbrace{tr(III_B\mathbf{I})}_{III_B*3} = 0$

$tr(\mathbf{B}^3) - tr(\mathbf{B})tr(\mathbf{B}^2) + \frac{1}{2}[(tr\mathbf{B})^2 - tr(\mathbf{B}^2)]tr\mathbf{B} = III_B * 3$

$tr(\mathbf{B}^3) - tr(\mathbf{B})tr(\mathbf{B}^2) + \underbrace{\frac{1}{2}tr(\mathbf{B})(tr\mathbf{B})^2}_{1/2(tr\mathbf{B})^3} - \frac{1}{2}tr(\mathbf{B})tr(\mathbf{B}^2) = III_B * 3$

$\frac{1}{3}\left[tr(\mathbf{B}^3) - \frac{3}{2}tr(\mathbf{B})tr(\mathbf{B}^2) + \frac{1}{2}(tr\mathbf{B})^3\right] = III_B$

$\frac{dIII_B}{d\mathbf{B}} = \frac{d\left(1/3tr(\mathbf{B}^3)\right)}{d\mathbf{B}} - \frac{d\left(1/2tr(\mathbf{B})tr(\mathbf{B}^2)\right)}{d\mathbf{B}} + \frac{d\left(1/6(tr\mathbf{B})^3\right)}{d\mathbf{B}}$

$= \frac{d\left(1/3tr(\mathbf{B}^3)\right)}{d\mathbf{B}} - \underbrace{\frac{d\left(1/2tr(\mathbf{B})\right)}{d\mathbf{B}} * tr(\mathbf{B}^2) - \frac{d(tr(\mathbf{B}^2))}{d\mathbf{B}} * \frac{1}{2}tr\mathbf{B}}_{\text{product rule}} + \underbrace{\frac{d(1/6(tr\mathbf{B})^3)}{d\mathbf{B}}}_{\frac{1}{6}\frac{d((tr\mathbf{B})^2 tr\mathbf{B})}{d\mathbf{B}}}$

$= \frac{1}{3}\frac{d(tr(\mathbf{B}^3))}{d\mathbf{B}} - \frac{1}{2}tr(\mathbf{B}^2)\underbrace{\frac{d(tr\mathbf{B})}{d\mathbf{B}}}_{\mathbf{I}} - \frac{1}{2}\underbrace{tr(\mathbf{B})}_{I_B}\underbrace{\frac{d(tr(\mathbf{B}^2))}{d\mathbf{B}}}_{2\mathbf{B}}$

$+ \overbrace{\frac{1}{6}tr(\mathbf{B})\underbrace{\frac{d((tr\mathbf{B})^2)}{d\mathbf{B}}}_{2tr(\mathbf{B})\frac{d(tr\mathbf{B})}{d\mathbf{B}}} + \frac{1}{6}(tr\mathbf{B})^2 * \underbrace{\frac{d(tr\mathbf{B})}{d\mathbf{B}}}_{\mathbf{I}}}^{\text{product rule}}$

$= \frac{1}{3}\underbrace{\frac{d(tr(\mathbf{B}^3))}{d\mathbf{B}}}_{\text{see below}} - \frac{1}{2}tr(\mathbf{B}^2)\mathbf{I} - \frac{1}{2}I_B(2\mathbf{B}) + 1/3(tr\mathbf{B})^2\mathbf{I} + 1/6(\underbrace{tr\mathbf{B}}_{I_B})^2\mathbf{I}$

where $\frac{d(tr(\mathbf{B}^3))}{d\mathbf{B}} = \frac{\partial[B_{kl}B_{lm}B_{mk}]}{\partial B_{ji}}\mathbf{e_i}\mathbf{e_j}$
$= \frac{\partial B_{kl}}{\partial B_{ji}}B_{lm}B_{mk}\mathbf{e_i}\mathbf{e_j} + B_{kl}\frac{\partial B_{lm}}{\partial B_{ji}}B_{mk}\mathbf{e_i}\mathbf{e_j} + B_{kl}B_{lm}\frac{\partial B_{mk}}{\partial B_{ji}}\mathbf{e_i}\mathbf{e_j}$
$= \delta_{kj}\delta_{li}B_{lm}B_{mk}\mathbf{e_i}\mathbf{e_j} + \delta_{lj}\delta_{mi}B_{kl}B_{mk}\mathbf{e_i}\mathbf{e_j} + \delta_{mj}\delta_{ki}B_{kl}B_{lm}\mathbf{e_i}\mathbf{e_j}$
$= B_{im}B_{mj}\mathbf{e_i}\mathbf{e_j} + B_{kj}B_{ik}\mathbf{e_i}\mathbf{e_j} + B_{il}B_{lj}\mathbf{e_i}\mathbf{e_j} = 3\mathbf{B}^2$

So,
$\frac{dIII_B}{d\mathbf{B}} = \frac{1}{3}(3\mathbf{B}^2) - \frac{1}{2}tr(\mathbf{B}^2)\mathbf{I} - \frac{1}{2}I_B(2\mathbf{B}) + \frac{1}{2}I_B^2\mathbf{I}$
$= \mathbf{B}^2 - I_B\mathbf{B} - \frac{1}{2}(tr(\mathbf{B}^2) - I_B^2)\mathbf{I}$

$$= \mathbf{B}^2 - I_B\mathbf{B} + II_B\mathbf{I} \ , \quad \text{Since } II_B = \frac{1}{2}\left(I_B^2 - tr(\mathbf{B}^2)\right) \tag{C.6}$$

C.3 Principal stretch constitutive relationship

Recall from an earlier chapter (example problem at the end of the section on Polar Decomposition) that we can form either the Right Stretch Tensor, $\mathbf{U}$, or the Left Stretch Tensor, $\mathbf{V}$, in their principal stress space by pre and post multiplying by the orthogonal tensor, $\mathbf{\Phi}$, where $\mathbf{\Phi}$ contains either the eigenvectors of the $\mathbf{U}$ or the eigenvectors of $\mathbf{V}$, as appropriate. Also recall that the eigen*values* of either tensor are the same, and their invariants are the same.

$[\mathbf{U}]_{\mathbf{n}} = [\mathbf{\Phi}]^T[\mathbf{U}][\mathbf{\Phi}]$

where $[\mathbf{\Phi}] = [\mathbf{\Phi}]_{\mathbf{U}} = \begin{bmatrix} (n_1)_{\lambda_1} & (n_1)_{\lambda_2} & (n_1)_{\lambda_3} \\ (n_2)_{\lambda_1} & (n_2)_{\lambda_2} & (n_2)_{\lambda_3} \\ (n_3)_{\lambda_1} & (n_3)_{\lambda_2} & (n_3)_{\lambda_3} \end{bmatrix}$

$[\mathbf{\Phi}]^T[\mathbf{U}][\mathbf{\Phi}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$[\mathbf{\Phi}]^T[\mathbf{E}][\mathbf{\Phi}] = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix} = \frac{1}{2}(\boldsymbol{\lambda}^2 - 1)$

$[\mathbf{\Phi}]^T[\hat{\boldsymbol{\sigma}}][\mathbf{\Phi}] = \frac{\partial\phi}{\partial[\mathbf{\Phi}]^T[\mathbf{E}][\mathbf{\Phi}]}$ (where $\hat{\boldsymbol{\sigma}} = \frac{\partial\phi}{\partial\mathbf{E}} = 2\frac{\partial\phi}{\partial\mathbf{C}} = 2\frac{\partial\phi}{\partial\mathbf{U}^2}$)

We know that the relationship between $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ is $\boldsymbol{\sigma} = \frac{1}{III_{\mathbf{U}}}\mathbf{F}\cdot\hat{\boldsymbol{\sigma}}\cdot\mathbf{F}^T$

Applying the operator $\mathbf{\Phi}^T\cdot\mathbf{A}\cdot\mathbf{\Phi}$ to all tensors in the above, we get:

$\mathbf{\Phi}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{\Phi} = \frac{1}{III_{\mathbf{U}}}\mathbf{\Phi}^T\cdot\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{\Phi}\frac{\partial\phi}{\partial\mathbf{\Phi}^T\cdot\mathbf{E}\cdot\mathbf{\Phi}}\mathbf{\Phi}^T\cdot\mathbf{U}^T\cdot\mathbf{R}^T\cdot\mathbf{\Phi}$

Pre-multiply $\mathbf{\Phi}^T\cdot\mathbf{R}^T\cdot\mathbf{\Phi}$ and post-multiply $\mathbf{\Phi}^T\cdot\mathbf{R}\cdot\mathbf{\Phi}$ to the above, and replace $\frac{\partial\phi}{\partial\mathbf{\Phi}^T\cdot\mathbf{E}\cdot\mathbf{\Phi}}$ with $2\frac{\partial\phi}{\partial\mathbf{\Phi}^T\cdot\mathbf{U}^2\cdot\mathbf{\Phi}}$:

$\mathbf{\Phi}^T\cdot\mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}\cdot\mathbf{\Phi} = \frac{2}{III_{\mathbf{U}}}\mathbf{\Phi}^T\cdot\mathbf{U}\cdot\mathbf{\Phi}\frac{\partial\phi}{\partial\mathbf{\Phi}^T\cdot\mathbf{U}^2\cdot\mathbf{\Phi}}\mathbf{\Phi}^T\cdot\mathbf{U}^T\cdot\mathbf{\Phi}$

We now note that $\mathbf{U}$ is symmetric, and $III_{\mathbf{U}}$ (which is the same as $III_{\mathbf{V}} = III_{\mathbf{F}} = III_{\mathbf{B}}^{1/2}$) can be expressed in terms of the principal stretches, as $\lambda_1\lambda_2\lambda_3$ (recall chapter 1). Furthermore, we can note that $\boldsymbol{\sigma}^* = \mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}$ from Appendix C.1, for example, where we took "$\mathbf{Q}$" to be equal to $\mathbf{R}^T$. We recall that $\boldsymbol{\sigma}^*$ was indeed found to be equal to some function $f(\mathbf{U})$.

$$\mathbf{\Phi}^T\cdot\mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}\cdot\mathbf{\Phi} = \frac{2}{\lambda_1\lambda_2\lambda_3}\mathbf{\Phi}^T\cdot\mathbf{U}\cdot\mathbf{\Phi}\frac{\partial\phi}{\partial\mathbf{\Phi}^T\cdot\mathbf{U}^2\cdot\mathbf{\Phi}}\mathbf{\Phi}^T\cdot\mathbf{U}\cdot\mathbf{\Phi} \qquad \text{(C.7)}$$

In order to arrive at our desired result, which expresses the principal values of the Cauchy stress, σ_i, as a function of the principal stretches, we'd like to pre multiply by $\mathbf{R}$ and post-multiplying by $\mathbf{R}^T$. This would give us the desired result on the left-hand-side.
Since $\underbrace{\mathbf{R}\cdot\mathbf{U}}_{\mathbf{F}} = \underbrace{\mathbf{V}\cdot\mathbf{R}}_{\mathbf{F}}$, we can see that $\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^T = \mathbf{V}$
In addition, $\mathbf{U}^2$ is $\mathbf{F}^T\cdot\mathbf{F} \longrightarrow \mathbf{R}\cdot\mathbf{U}^2\cdot\mathbf{R}^T = \mathbf{R}\cdot\underbrace{\mathbf{R}^T\cdot\mathbf{V}^2\cdot\mathbf{R}}_{\mathbf{F}^T\cdot\mathbf{F}}\cdot\mathbf{R}^T = \mathbf{V}^2$
So, pre multiplying by $\mathbf{R}$ and post-multiplying by $\mathbf{R}^T$ on eq. (C.7), yields:

$$\mathbf{\Phi}^T\boldsymbol{\sigma}\mathbf{\Phi} = \frac{2}{\lambda_1\lambda_2\lambda_3}\mathbf{\Phi}^T\cdot\mathbf{V}\cdot\mathbf{\Phi}\frac{\partial\phi}{\partial\mathbf{\Phi}^T\cdot\mathbf{V}^2\cdot\mathbf{\Phi}}\mathbf{\Phi}^T\cdot\mathbf{V}\cdot\mathbf{\Phi} \tag{C.8}$$

To arrive at our final result, we need to make a few more observations. Namely, we observe that all $\mathbf{\Phi}$ that are present in eq. (C.8) and that may have been up to this point assumed to represent $\mathbf{\Phi_U}$ can easily be replaced by $\mathbf{\Phi_V}$ without affecting any of the algebra that we've already done. $\mathbf{\Phi}$ can be any orthogonal tensor at this point. In fact, we've seen eq. (C.8) before, but without the presence of $\mathbf{\Phi}$ (*i.e.* $\mathbf{\Phi} = \mathbf{I}$ yields eq. (6.2)). By taking $\mathbf{\Phi} = \mathbf{\Phi_V}$, we can now replace $\mathbf{\Phi}^T\cdot\mathbf{V}\cdot\mathbf{\Phi}$ with $\boldsymbol{\lambda}$ and $\mathbf{\Phi}^T\cdot\mathbf{V}^2\cdot\mathbf{\Phi}$ with $\boldsymbol{\lambda}^2$.

In addition, we note that $\frac{\partial\phi}{\partial\boldsymbol{\lambda}} = \frac{\partial\phi}{\partial f(\boldsymbol{\lambda})}\frac{\partial f(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}} \rightarrow \frac{\partial\phi}{\partial f(\boldsymbol{\lambda})} = \frac{1}{\frac{\partial f(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}}}\frac{\partial\phi}{\partial\lambda}$

Note that $\frac{\partial\phi}{\partial\boldsymbol{\lambda}}$ is taken as the partial derivative here (as opposed to a total derivative like $\frac{d\phi}{d\mathbf{B}}$), since we will see shortly that the form of our strain energy density of interest (Ogden) is a function of λ rather than the strain invariants. Also note that because $\boldsymbol{\lambda}$ is diagonal, we are skipping the formal proof, here, for $\frac{\partial f(\boldsymbol{\lambda})}{\partial\boldsymbol{\lambda}} = \frac{\partial\boldsymbol{\lambda}^2}{\partial\boldsymbol{\lambda}} = 2\boldsymbol{\lambda}$ as well as the proof of $\frac{\partial\phi}{\partial\boldsymbol{\lambda}} = \frac{\partial\phi}{\partial\lambda}\mathbf{I} = \frac{\partial\phi}{\partial\lambda_i}\delta_{ij}\mathbf{e_i}\mathbf{e_j}$, along with the more obvious property of a diagonal tensor: $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = \lambda^2\mathbf{I} = \lambda_i^2\delta_{ij}\mathbf{e_i}\mathbf{e_j}$

With these substitutions, eq. (C.8) becomes

$$\sigma_i = \frac{2}{\lambda_1\lambda_2\lambda_3}\lambda_i^2\left(\frac{1}{2\lambda_i}\frac{\partial\phi}{\partial\lambda_i}\right) = \frac{1}{\underbrace{\lambda_j\lambda_k}_{\text{no sum}}}\frac{\partial\phi}{\partial\lambda_i} \tag{C.9}$$

C.4 Tabulated hyperelastic model

We start with the Ogden model - *viz*,
$\sigma_i = \sum_{s=1}^{m} \frac{\mu_s}{III_{\mathbf{V}}} \left[\lambda_i^{*\alpha_s} - \sum_{n=1}^{3} \frac{\lambda_n^{*\alpha_s}}{3} \right] + K \frac{III_{\mathbf{V}}-1}{III_{\mathbf{V}}}$

Let's define a function:

$$f_0(\lambda) = \sum_{s=1}^{m} \mu_s \lambda^{*\alpha_s} \tag{C.10}$$

Substituting yields:

$$\sigma_i = \frac{1}{III_{\mathbf{V}}} \left(f_0(\lambda_i) - \frac{1}{3} \sum_{n=1}^{3} f_0(\lambda_n) \right) + K \frac{III_{\mathbf{V}} - 1}{III_{\mathbf{V}}} \tag{C.11}$$

Recall from the first Mooney-Rivlin example, that for an incompressible material under uniaxial test conditions, $\lambda_j^* \approx \lambda_k^* \approx \lambda_i^{*-1/2}$ $(\lambda_j \approx \lambda_k \approx \lambda_i^{-1/2})$, where the subscripts j and k refer to the two coordinate directions perpendicular to i, just as before.

The engineering stress, which would be commonly retrieved from a uniaxial test, is the nominal stress for a hyperelastic material (recall the formula for nominal stress, namely, $\boldsymbol{\sigma}^0 = III_{\mathbf{V}} \mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$)

$$\boldsymbol{\sigma}^0 = \lambda_i \lambda_j \lambda_k \begin{bmatrix} \frac{1}{\lambda_i} & 0 & 0 \\ 0 & \lambda_i^{1/2} & 0 \\ 0 & 0 & \lambda_i^{1/2} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\longrightarrow \sigma_{11}^0 = \underbrace{\lambda_j \lambda_k}_{\text{no sum}} \sigma_{11}$$

eq. (C.11), for uniaxial load under the Ogden model, will be expressed as $\sigma_1 = \sigma_{11} =$"$\sigma(\lambda_i)$". So, $\sigma(\lambda_i)$ is a particular value of longitudinal Cauchy stress under uniaxial loading, which corresponds to a particular value of the longitudinal stretch, λ_i, under the Ogden model.

We'll introduce a new variable, ϵ_{0i}, which is the engineering strain in a uniaxial test - *i.e.* $\epsilon_{0i} = \lambda_i - 1$ for a hyperelastic material if λ_i is some longitudinal stretch that occurred during the uniaxial test. Presumably, we have an experimental curve of uniaxial engineering stress, which we will from now on call σ_0, as a function of the longitudinal engineering strain $\big(i.e.\ \sigma_0(\epsilon_{0i})\big)$.

With our new notation, we can define σ_{11}^0 as follows:

$$\sigma_0(\epsilon_{0i}) = \sigma_0(\lambda_i - 1) = \underbrace{\lambda_j\lambda_k}_{\text{no sum}}\,\sigma(\lambda_i)$$

Now summing only on repeated indices unless otherwise noted, we can march through the derivation of this tabulated model. First, we observe eq. (C.10), and note that:

$$f_0(\lambda_i) = \sum_{s=1}^{m} \mu_s \lambda_i^{*\alpha_s} \tag{C.12}$$

Additionally, we notice the term $\frac{1}{3}\sum_{n=1}^{3} f_0(\lambda_n)$ in eq. (C.11), and are thus interested in the following calculation:

$$\sum_{n=1}^{3} f_0(\lambda_n) = f_0(\lambda_i) + f_0(\lambda_j) + f_0(\lambda_k) = \left[\sum_{s=1}^{m} \mu_s \lambda_i^{*\alpha_s} + 2\sum_{s=1}^{m} \mu_s \lambda_i^{-\frac{\alpha_s}{2}}\right] \tag{C.13}$$

where $f_0(\lambda_j)$ and $f_0(\lambda_k)$ were taken to be $f_0(\lambda_i^{-1/2})$

Substituting eq. (C.12) and eq. (C.13) into the original stress equation (e.x. eq. (C.11)) gives the following result:

$$\sigma_0(\lambda_i - 1) = \underbrace{\lambda_i\lambda_k}_{\text{no sum}}\,\sigma(\lambda_i) = \frac{\overbrace{\lambda_k\lambda_j}^{\text{no sum}}}{III_{\mathbf{V}}}\left(\tfrac{2}{3}f_0(\lambda_i) - \tfrac{2}{3}f_0(\lambda_i^{-1/2}) + K\tfrac{III_{\mathbf{V}}-1}{III_{\mathbf{V}}}III_{\mathbf{V}}\right)$$
$$= \tfrac{1}{\lambda_i}\left(\tfrac{2}{3}f_0(\lambda_i) - \tfrac{2}{3}f_0(\lambda_i^{-1/2}) - pIII_{\mathbf{V}}\right)$$
$$\lambda_i\sigma_0(\lambda_i - 1) + p = \frac{2}{3}f_0(\lambda_i) - \frac{2}{3}f_0(\lambda_i^{-1/2}) \tag{C.14}$$

$III_{\mathbf{V}}$ was eliminated from eq. (C.14) since we are going to limit our discussion to incompressible materials only.

Note that "p" is really a hydrostatic term that depends on "K," which in our case is arbitrary. Simply striking the term would not stay true to the Ogden function and could cause undesirable behavior. However, we can eliminate the term through consideration of boundary conditions.

For uniaxial stress, $\sigma(\lambda_j) = \sigma(\lambda_k) = \sigma(\lambda_i^{-1/2}) = 0$.

Eq. (C.11) yields:

$$0 = \frac{1}{3} f_0(\lambda_i^{-1/2}) - \frac{1}{3} f_0(\lambda_i) + \underbrace{K \frac{III_{\mathbf{V}} - 1}{III_{\mathbf{V}}}}_{p} \tag{C.15}$$

In eq. (C.15), we find that p must equal:

$$p = \frac{1}{3} f_0(\lambda_i) - \frac{1}{3} f_0(\lambda_i^{-1/2}) \tag{C.16}$$

Eq. (C.16) $\longrightarrow$ eq. (C.14) yields:

$$\lambda_i \sigma_0(\lambda_i - 1) = f_0(\lambda_i) - f_0(\lambda_i^{-1/2}) \tag{C.17}$$

We can substitute consecutive values of the principal stretch into eq. (C.17).

i.e.

$\lambda_i^{-1/2} \sigma_0(\lambda_i^{-1/2} - 1) = f_0(\lambda_i^{-1/2}) - f_0(\lambda_i^{1/4})$
$\lambda_i^{1/4} \sigma_0(\lambda_i^{1/4} - 1) = f_0(\lambda_i^{1/4}) - f_0(\lambda_i^{-1/8})$
.
.
etc.

In general,

$$\lambda_i^{(-1/2)^{x-1}} \sigma_0 \left(\lambda_i^{(-1/2)^{x-1}} - 1 \right) = f_0 \left(\lambda_i^{(-1/2)^{x-1}} \right) - f_0 \left(\lambda_i^{(-1/2)^{x}} \right) \tag{C.18}$$

Since $\lim_{x \to \infty} f_0 \left(\lambda_i^{(-1/2)^x} \right) = f_0(1)$, where $f_0(1) = \sum_{s=1}^{m} \mu_s$,

we get: $\sum_{x=1}^{\infty} \lambda_i^{(-1/2)^{x-1}} \sigma_0 \left(\lambda_i^{(-1/2)^{x-1}} - 1 \right) = f_0(\lambda_i) - f_0(1)$

where all terms on the right hand side cancel, except for the first and last.

So, $f_0(\lambda_i) = f_0(1) + \lambda_i \sigma_0(\lambda_i - 1) + \lambda_i^{-1/2}\sigma_0(\lambda_i^{-1/2} - 1) + \lambda_i^{1/4}\sigma_0(\lambda_i^{1/4} - 1) + ...$

Writing this as concisely as possible:

$$f_0(\lambda_i) = f(1) + \sum_{x=0}^{\infty} \lambda_i^{(-1/2)^x} \sigma_0 \left(\lambda_i^{(-1/2)^x} - 1\right) \tag{C.19}$$

We can now substitute eq. (C.19) into eq. (C.11). Since $f(1)$ is a constant, we can see that it doesn't affect the stress, σ_i, since $f(1) - 1/3 \sum_{n=1}^{3} f(1) = 0$.

To complete our discussion, $f_0(\lambda_i)$ and σ_0 will be written a final time, in their final form:

$$f_0(\lambda_i) = \sum_{x=0}^{\infty} \lambda_i^{(-1/2)^x} \sigma_0 \left(\lambda_i^{(-1/2)^x} - 1\right) \tag{C.20}$$

$$\sigma_i = \frac{1}{III_{\mathbf{V}}} \left(f_0(\lambda_i) - \frac{1}{3} \sum_{n=1}^{3} f_0(\lambda_n) \right) + K \frac{III_{\mathbf{V}} - 1}{III_{\mathbf{V}}} \tag{C.21}$$

The way that this model works is described in a previous section. The introduction of $f_0(\lambda_i)$, which eliminates the material constants from the Ogden model (*i.e.* eq. (C.11)) was important, but it was the step from eq. (C.18) to eq. (C.19) that enables this "tabulated" method to work as desired. The particular pattern that was recognized by the aforementioned researchers that developed this "tabulated" method [17], which is expressed in eq. (C.18), along with the observation that summing the right-hand-side of eq. (C.18) cancels most of the terms, were really the key insights to isolate the $f_0(\lambda_i)$ term.

Appendix D

Chapter 7 derivations

D.1 Jaumann rate in infinitesimal elasticity

Let's define $\boldsymbol{\sigma}$ as the infinitesimal stress tensor that we want to obtain, and $\hat{\boldsymbol{\sigma}}$ as the infinitesimal stress tensor in material coordinates (here, $\boldsymbol{\sigma}$ is in spatial coordinates, hence the use of the variable "σ" that has been previously reserved for the Cauchy stress). With these definitions, consider Fig. D.1.

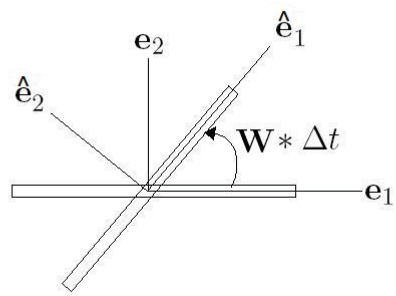

Figure D.1: Rotating body without shear

With respect to the stress measures depicted in Fig. D.1, let's take the time derivative:
i.e. Let us start with $\sigma_{ij}\mathbf{e_i}\mathbf{e_j} = \hat{\sigma}_{ij}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}$ since these two measures of stress are merely transformations of each other in linear infinitesimal elasticity, and are identical at time t=0. Now, take the time derivative of both sides of the equality.

$\dot{\boldsymbol{\sigma}} = \dot{\sigma}_{ij}\mathbf{e_i}\mathbf{e_j} + \sigma_{ij}\cancel{\dot{\mathbf{e}}_\mathbf{i}}\mathbf{e_j} + \sigma_{ij}\mathbf{e_i}\cancel{\dot{\mathbf{e}}_\mathbf{j}} = \dot{\hat{\sigma}}_{ij}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j} + \hat{\sigma}_{ij}\dot{\hat{\mathbf{e}}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j} + \hat{\sigma}_{ij}\hat{\mathbf{e}}_\mathbf{i}\dot{\hat{\mathbf{e}}}_\mathbf{j}$,
where the slashed terms are zero. $\boxed{\text{Also note: } \dot{\hat{\mathbf{e}}}_\mathbf{i} = W_{ki}\hat{\mathbf{e}}_\mathbf{k}}$
$\longrightarrow \dot{\boldsymbol{\sigma}} = \dot{\hat{\sigma}}_{ij}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}+\hat{\sigma}_{ij}W_{ki}\hat{\mathbf{e}}_\mathbf{k}\hat{\mathbf{e}}_\mathbf{j}+\hat{\sigma}_{ij}\hat{\mathbf{e}}_\mathbf{i}W_{kj}\hat{\mathbf{e}}_\mathbf{k} = \dot{\hat{\sigma}}_{ij}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}+\hat{\sigma}_{kj}W_{ik}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}+\hat{\sigma}_{ik}\hat{\mathbf{e}}_\mathbf{i}W_{jk}\hat{\mathbf{e}}_\mathbf{j}$
$= \left[\dot{\hat{\sigma}}_{ij} + \hat{\sigma}_{kj}W_{ik} + \hat{\sigma}_{ik}W_{jk}\right]\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}$

$$\dot{\sigma}_{ij}\mathbf{e_i}\mathbf{e_j} = \left[\dot{\hat{\sigma}}_{ij} + \hat{\sigma}_{kj}W_{ik} + \hat{\sigma}_{ik}W_{jk}\right]\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j} \tag{D.1}$$

Recall that at the beginning of the derivation, we noted that $\sigma_{ij}\mathbf{e_i}\mathbf{e_j} = \hat{\sigma}_{ij}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}$. In addition, note that $\dot{\hat{\sigma}}_{ij}\hat{\mathbf{e}}_\mathbf{i}\hat{\mathbf{e}}_\mathbf{j}$ needs to be transformed to the spatial bases, which can be accomplished using the transformation $\mathbf{F}\cdot\dot{\hat{\boldsymbol{\sigma}}}\cdot\mathbf{F}^T$. Thus, eq. (D.1) becomes:

$$\dot{\boldsymbol{\sigma}} = \mathbf{F}\cdot\dot{\hat{\boldsymbol{\sigma}}}\cdot\mathbf{F}^T + \mathbf{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{W}^T \tag{D.2}$$

Recognizing $\mathbf{F}\cdot\dot{\hat{\boldsymbol{\sigma}}}\cdot\mathbf{F}^T$ to be the Jaumann rate, with $det\mathbf{F}$ taken to be approximately unity for infinitesimal deformations, we arrive at the desired result:

$$\dot{\boldsymbol{\sigma}} = \mathring{\boldsymbol{\sigma}} + \mathbf{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{W}^T \tag{D.3}$$

Eq. (D.3) is exactly the same as our previous expression for the Jaumann rate from a previous section. It is derived in a different, more general, way in Appendix D.2.

D.2 Truesdell and Jaumann rates

From the formula for the 2^{nd} Piola-Kirchhoff stress, we know that:

$$\boldsymbol{\sigma} = \frac{1}{det\mathbf{F}}\mathbf{F} \cdot \hat{\boldsymbol{\sigma}} \cdot \mathbf{F}^T \tag{D.4}$$

Before we take the time derivative of eq. (D.4), note that:

$$\frac{d}{dt}\left(det\mathbf{F}\right)^{-1} = -\left(det\mathbf{F}\right)^{-2} * \dot{\overline{det\mathbf{F}}} = \frac{-(\cancel{det\mathbf{F}})tr\mathbf{D}}{(det\mathbf{F})^{\cancel{2}}} \tag{D.5}$$

Note that in eq. (D.5), we used the equality $\dot{\overline{det\mathbf{F}}} = (det\mathbf{F})tr\mathbf{D}$, which was proven in the derivation of the hypoelastic constitutive relationship involving the Jaumann rate (from a previous chapter).

Now, taking the time derivative of eq. (D.4), we have:

$$\dot{\boldsymbol{\sigma}} = -\frac{tr\mathbf{D}}{det\mathbf{F}}\mathbf{F} \cdot \hat{\boldsymbol{\sigma}} \cdot \mathbf{F}^T + \frac{1}{det\mathbf{F}}\dot{\mathbf{F}} \cdot \hat{\boldsymbol{\sigma}} \cdot \mathbf{F}^T + \frac{1}{det\mathbf{F}}\mathbf{F} \cdot \dot{\hat{\boldsymbol{\sigma}}} \cdot \mathbf{F}^T + \frac{1}{det\mathbf{F}}\mathbf{F} \cdot \hat{\boldsymbol{\sigma}} \cdot \dot{\mathbf{F}}^T$$

Now, we know $\hat{\boldsymbol{\sigma}}$ in terms of $\boldsymbol{\sigma}$ from our Piola-Kirchhoff relationship derived in a previous section. Substituting, we get:

$$\begin{aligned}\dot{\boldsymbol{\sigma}} = &-\frac{tr\mathbf{D}}{\cancel{det\mathbf{F}}}\cancel{\mathbf{F}} \cdot (\cancel{det\mathbf{F}})\cancel{\mathbf{F}^{-1}} \cdot \boldsymbol{\sigma} \cdot \cancel{\mathbf{F}^{-T}} \cdot \cancel{\mathbf{F}^T} + \frac{1}{\cancel{det\mathbf{F}}}\dot{\mathbf{F}} \cdot (\cancel{det\mathbf{F}})\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \cancel{\mathbf{F}^{-T}} \cdot \cancel{\mathbf{F}^T}\\ &+ \frac{1}{det\mathbf{F}}\mathbf{F} \cdot \dot{\hat{\boldsymbol{\sigma}}} \cdot \mathbf{F}^T + \frac{1}{\cancel{det\mathbf{F}}}\cancel{\mathbf{F}} \cdot (\cancel{det\mathbf{F}})\cancel{\mathbf{F}^{-1}} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \cdot \dot{\mathbf{F}}^T\end{aligned}$$

Thus, we arrive at the desired result:

$$\dot{\boldsymbol{\sigma}} = -tr(\mathbf{D})\boldsymbol{\sigma} + \mathbf{L} \cdot \boldsymbol{\sigma} + \overset{\triangledown}{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \cdot \mathbf{L}^T$$

Or,

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{L} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{L}^T + tr(\mathbf{D})\boldsymbol{\sigma} \tag{D.6}$$

Throwing all $\mathbf{D}$ terms out except for where it appears in the constitutive expression, we can arrive at the Jaumann expression (e.x. eq. (D.3)):

$$\dot{\boldsymbol{\sigma}} = \mathring{\boldsymbol{\sigma}} + \mathbf{W} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{W}^T \tag{D.7}$$